WORKSHOP ON "INTERGRAL p-ADIC HODGE THEORY"

Let C be the completion of an algebraic closure of $\mathbb{Q}_p$, $\mathcal{O}_C$ be its integral ring, and $k \cong \overline{\mathbb{F}}_p$ be the residue field. Let $A_{\text{inf}} = W(\mathcal{O}_C^\flat)$. Bhatt, Morrow and Scholze [BMS16] developed recently a new cohomology theory $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$ for a proper smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$. The cohomology groups $\mathrm{H}^i_{A_{\text{inf}}}(\mathfrak{X})$ are finitely presented A_{inf}-modules equipped with a semi-linear Frobenius action satisfying certain properties. Such objects are called Breuil–Kisin–Fargues modules, and they can be viewed as analogues of Drinfeld's shtukas with one leg in mixed characteristic. This A_{inf}-cohomology can be practically viewed as a universal p-adic cohomology theory in the sense that it specialized to all other known p-adic cohomologies such as crystalline cohomology for the special fiber $\mathfrak{X}_k$, the p-adic étale cohomology for adic generic fiber $\mathfrak{X}_C^{\text{ad}}$, and the de Rham-Witt cohomology for $\mathfrak{X}$. More importantly, this new cohomology theory preserves the information on torsions in various p-adic cohomology. It can be used to establish a comparison theorem between p-adic étale cohomology on the generic fiber $\mathfrak{X}_C^{\text{ad}}$ and the crystalline cohomology on the special fiber $\mathfrak{X}_k$ with integral coefficients. In his report for ICM 2018 [Sch17], Scholze considers this as a first step towards an explicit universal cohomology theory for algebraic varieties over $\mathbb{Q}$ that plays practically the same role as the conjectured motives.

The aim of this workshop is to understand the construction of this A_{inf}-cohomology theory and its applications to p-adic integral Hodge theory. Basic requisites include Scholze's theory of perfectoid spaces, and the pro-étale site for rigid analytic varieties.

0.1. **Introduction.** Give an introduction to the main results of [BMS16] and discuss some interesting examples. Section 1 and 2 of [BMS16].

0.2. **Algebraic properties of A_{inf}-modules.** Section 4.2 of [BMS16]. Structure of finite presented modules over A_{inf}. The equivalence between the category of vector bundles over $\mathrm{Spec}(A_{\text{inf}})$ and $\mathrm{Spec}(A_{\text{inf}})\backslash\{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of A_{inf}.

0.3. **Breuil–Kisin modules and Breuil–Kisin–Fargues modules.** Section 4.1, 4.3, 4.4 of [BMS16]. Explain Fargues' theorem [BMS16, Theorem 4.28] on the equivalence between the category Breuil–Kisin–Fargues modules and that of pairs (T, Ξ), where T is a finite free $\mathbb{Z}_p$-module and Ξ is a B_{dR}^+-lattice in $T \otimes_{\mathbb{Z}_p} B_{\text{dR}}$.

0.4. **Review of Scholze's Pro-étale site.** Give a review on Scholze's pro-étale site on rigid analytic varieties and Scholze's primitive comparison Theorem [Sch13, Corollary 5.11].

0.5. **De Rham comparison theorem for rigid analytic varieties over C.** For a rigid analytic variety X/C, introduce the cohomology group $\mathrm{H}^i_{\mathrm{crys}}(X/B^+_{\mathrm{dR}})$, explain its comparison with étale cohomology and usual de Rham cohomology if X descends to a finite extension of $\mathbb{Q}_p$. [BMS16, Section 12]

0.6. **$L\eta$-operator.** Construction of $L\eta$-operator on the derived category of abelian categories. Basic properties. [BMS16, Section 6] and Lemma 8.11.

0.7. **The complexe $\tilde{\Omega}_{\mathfrak{X}}$ and local computations.** Define the complex $\tilde{\Omega}_{\mathfrak{X}}$ and compute it in local situations. Show that $\tilde{\Omega}_{\mathfrak{X}}$ is a perfect complex of $\mathcal{O}_{\mathfrak{X}}$-modules, and one has an isomorphism $\mathrm{H}^i(\tilde{\Omega}_{\mathfrak{X}}) \cong \Omega^{i,cont}_{\mathfrak{X}/\mathcal{O}}\{-i\}$ (Theorem 8.3) [BMS16, Section 7 and 8].

0.8. **Relative De Rham-Witt complex.** Give a review on the relative de Rham-Witt complex and its relation with crystalline cohomology. [BMS16, Section 10] and [LZ04, Theorem 3.5].

0.9. **The complexe $A\Omega_{\mathfrak{X}}$.** Introduce the complex $A\Omega_{\mathfrak{X}}$, explain its basic properties. [BMS16, Section 9]

0.10. **Comparison with de Rham-Witt complex.** Compare the complex $A\Omega_{\mathfrak{X}}$ with the de Rham-Witt complex. [BMS16, Section 11].

0.11. **Comparison with crystalline cohomology over A_{crys}.** Prove that we can recover the crystalline cohomology of $\mathfrak{X}$ over A_{crys} from the complex $A\Omega_{\mathfrak{X}}$ [BMS16, Theorem 12.1].

References

[BMS16] B. Bhatt, M.Morrow, P. Scholze, Intergral p-adic Hodge theory, preprint in 2016, `arXiv:1602.03148`

[LZ04] Langer, A., and Zink, T. De Rham-Witt cohomology for a proper and smooth morphism. J. Inst. Math. Jussieu 3, 2 (2004), 231-314.

[Sch13] P. Scholze, p-adic Hodge theory for rigid-analytic varieties, Forum of Mathematics, Pi, 1, e1, 2013.

[Sch17] P. Scholze, p-adic geometry, talk for ICM 2018, available at author's homepage.