Mini course
"Introduction to holomorphic vector bundles on compact Riemann surfaces"
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The theory of holomorphic vector bundles on complex manifolds has become a central field of complex geometry. Classically, the theory of holomorphic line bundles and their global sections on a compact Riemann surface X may be viewed as the theory of meromorphic functions on X with certain conditions on their zeroes and poles. It governs the projective geometry of X.

Holomorphic vector bundles of higher rank arise, e.g., as normal bundles for complex embeddings $X \hookrightarrow \mathbb{P}^n(\mathbb{C})$. They can also be constructed from representations $\varrho \colon \pi_1(X) \longrightarrow \mathrm{GL}_r(\mathbb{C})$ of the fundamental group of X.

Vector bundles on a compact Riemann surface X have two discrete, topological invariants, the rank r and the degree d. If $r = 1$ and d is fixed, there is a nice complex manifold parameterizing all line bundles of degree d on X, denoted by $\mathrm{Jac}^d(X)$. If $r \geq 2$ and degree d is fixed, it is no longer possible to parameterize all vector bundles of rank r and degree d by a reasonable complex space. Here, one has to distinguish between unstable and semistable vector bundles, and there are different approaches to the classification of the bundles in the two classes.

The moduli space $\mathcal{M}_{r/d}(X)$ of semistable vector bundles of given rank r and degree d is an interesting complex space which has been thoroughly studied in complex algebraic geometry.

In the lectures, we will have a glimpse at the topics we have just mentioned.

We will mainly follow the classical text [3] by R.C. Gunning and complement it by an account on the theory of moduli spaces of semistable vector bundles, based on [6] and [4].

As a prerequisite, a basic knowledge of complex geometry and compact Riemann surfaces as offered by the books [1] and [2] will be more than sufficient.

Lecture 1.
Review of divisors and line bundles

We will first give the definition of a holomorphic vector bundle on a complex manifold and briefly discuss the description of vector bundles in terms of cocycles. This will be the basis for many of the subsequent arguments.

Vector bundles of rank one are called line bundles. We will review the connection between line bundles, divisors, maps to projective spaces, etc., in the case of compact Riemann surfaces. We will also speak about the Riemann–Roch theorem.

Lecture 2.
Motivation and basic tools

We will start with a brief discussion motivating the study of vector bundles on compact Riemann surfaces.

Afterwards, we will introduce some basic notions and techniques for investigating holomorphic vector bundles.

We will state the Riemann–Roch theorem for vector bundles and explain how it reduces to the Riemann–Roch theorem for line bundles.

Lecture 3.
Representations of the fundamental group and Weil's theorem

As discussed in Lecture 2, a representation $\varrho \colon \pi_1(X) \longrightarrow \mathrm{GL}_r(\mathbb{C})$ gives rise to a holomorphic vector bundle E_ϱ of rank r and degree 0 on the compact Riemann surface X.

A classical theorem of Weil [7] characterizes the holomorphic vector bundles on X which may be obtained in this way. We will try to present some ideas and steps in the proof of Weil's theorem.

Lecture 4.
Unstable bundles of rank two and extensions

We will discuss the Harder–Narasimhan filtration of a holomorphic vector bundle. For an unstable[1] vector bundle E of rank two, this yields the existence of a unique destabilizing line bundle, so that E may be presented in a canonical way as the extension of two line bundles.

We will explain the relationship between the classification of extensions and the classification of vector bundles in general and apply the results to the classification of unstable vector bundles of rank two.

Lecture 5.
Semistable vector bundles

When it comes to the classification of vector bundles, the concept of a semistable vector bundle is the central one. Semistable vector bundles of fixed rank and degree possess a so-called moduli space. This is an interesting projective algebraic variety in its own right. In order to explain the points of the moduli space, we will explain the notion of S-equivalence.

Picking up the theme of representations of the fundamental group again, we will briefly speak about the theorem of Narasimhan and Seshadri [5].

References

[1] O. Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer-Verlag, New York, 1991, viii+254 pp. ISBN: 0-387-90617-7.

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[1]Or, rather an unsemistable one.

[2] R.C. Gunning, *Lectures on Riemann surfaces*, Princeton Mathematical Notes, Princeton University Press, Princeton, N.J., 1966, iv+254 pp.

[3] R.C. Gunning, *Lectures on vector bundles over Riemann surfaces*, University of Tokyo Press, Tokyo; Princeton University Press, Princeton, N.J., 1967, v+243 pp.

[4] J. Le Potier, *Lectures on vector bundles*, Cambridge Studies in Advanced Mathematics, vol. 54, Cambridge University Press, Cambridge, 1997, viii+251 pp. ISBN: 0-521-48182-1.

[5] M.S. Narasimhan, C.S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540-67.

[6] P.E. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 51, Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978, vi+183 pp. ISBN: 0-387-08851-2.

[7] A. Weil, *Généralisation des fonctions abéliennes*, J. Math. pur. appl. (9) **17** (1938), 47-87.