DIOPHANTINE GEOMETRY ON CURVES OVER FUNCTION FIELDS

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These are extended notes of my five hours lecture at the *Winter School on Algebraic Curves, Riemann Surfaces and Moduli Spaces* held at the Morningside Center of Mathematics, CAS from march 4th to 8th 2019.

We explain the main steps of the proof of three cornerstone theorems in the diophantine geometry of smooth projective curves over a function field in one variable in characteristic zero:

Let F be the function field of a smooth projective curve B defined over $\mathbf{C}$.

– We prove that every X_F curve of genus zero defined over F has infinitely many F–rational points and therefore it is isomorphic, over F, to $\mathbf{P}^1$.

– We introduce then the important concept of isotrivality for curves defined over F: Isotrivial curves are those which, after a finite extension of F, are isomorphic to curves defined over $\mathbf{C}$ (which is naturally a subfield of F). In particular we will prove an important isotriviality criterion as a consequence of a classical Theorem by De Franchis (which is proved in the notes).

– We prove that a smooth projective curve X_F of genus one, equipped with a rational point, has a natural structure of a commutative group variety. If the curve is *not* isotrivial, then we will prove that this group is finitely generated. This is known as the Mordell Weil Theorem.

– We prove that a *non* isotrivial smooth projective curve X_F of genus at least two has only finitely many F–rational points.

We tried to keep the notes as much as self contained as possible. Nevertheless, the proof of all these theorems require properties of the algebraic geometry of curves and surfaces. Most of them are quite standard, for instance they are treated in the classical book by Hartshorne [10] or, for the analytic point of view, the book by Griffiths and Harris [7]. We provide proofs of statements which are specific to these notes.

Very good references which cover many of the topic treated on this course and where one can find the proof of almost all the quoted properties are [11] and [2].

There are few standard main theorems whose proof requires a little bit deeper knowledge of algebraic geometry:

–The semi stable reduction Theorem: Theorem 4.4, whose proof is not very difficult in characteristic zero (but it requires a proof) and a proof may be found for instance in [9].

– The existence of parameter spaces for morphisms and isomorphisms. These are described for instance in [3], in [8] and [5] (for parameter spaces of isomorphisms).

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– The basic theory of foliations : any standard book on ordinary differential equations over the complex numbers would be sufficient to cover the facts used in these lectures.

The topic of thiese lectures may also be seen as a ”motivating topic” of part of algebraic geometry: most of the classical theorems of the algebraic geometry of curves and surfaces are used to obtain some relatively explicit theorems in diophantine geometry. These lectures do not pretend any originality: most of the statements are very classical, the only contribution here is to provide a diophantine geometry motivation and all the mistakes and inaccuracies.

Contents

1. Lecture One

One of the main objectives in mathematics is solving algebraic equations. Given a system of algebraic equations $G(X) = 0$ defined over a ring A, may we know if there are solutions of it on A? and, in case, may we explicitly find them? Of course the prototype of such a ring is $\mathbf{Z}$ or rings finite over it, but the theorem of Matiyasevich tells us that we cannot find a general method to answer to the question in this situation.

Never the less one can try to see if it is possible to answer to the question for some class of systems of algebraic equations.

Since long time, we know that exists a similarity within the following three rings:

$$\mathbf{Z}; \qquad \mathbf{C}[t]; \qquad \mathbb{F}_q[t]. \tag{1.1}$$

– They are all P.I.D.
– They have a unique prime ideal (0) which is dense in the Zariski topology and all the other prime ideals are maximal.
– There is also another analogy called "the Product formula" which, for the last two rings is equivalent to the fact that a rational function on $\mathbf{P}^1$ has as many zeros as many poles (counted with multiplicity).

Essentially all the technics one can develop to study the system of polynomial equations over $\mathbf{Z}$ may be developed also for studying the same theory over $\mathbf{C}[t]$ or $\mathbb{F}_q[t]$. But on the rings $k[t]$ (with k an algebraically closed field) we can use tools coming from projective geometry and, in the case of $\mathbf{C}[t]$, from topology. Thus one can hope that the study of the diophantine equations may be easier to solve over these rings then over $\mathbf{Z}$. Consequently before we attack the theory of systems of polynomial equations over $\mathbf{Z}$, we can try to study the same theory over $\mathbf{C}[t]$ or $\mathbb{F}_q[t]$. In most of the cases, a statement which is false in the theory of polynomial equations over $k[t]$ is also false (often for the same reasons) when restated in the theory of polynomial equations over $\mathbf{Z}$.

The main topic of *Diophantine geometry* is attempt to answer to the following meta question:
Given a system of polynomial equations $G(X) = 0$ over a ring R, can we understand the structure of the set of solutions of it from the properties of R and the *geometric* properties of the algebraic variety $\mathbf{V}(G)$?
"Understanding the structure of the set of solutions" means being able to answer to some of these questions:
– Under which geometric conditions this set of solutions is finite?
– Under which conditions this set is or is not Zariski dense in $\mathbf{V}(G)$?
– Is there some particular structure on this set? (for instance a group structure, and in this case what kind of group?)

Let K be the field of fraction of the involved ring R. By "geometric properties of $\mathbf{V}(G)$" we mean the properties of the variety $\mathbf{V}(G)$ over the algebraic closure of K.

In these lectures we will concentrate on the Diophantine Geometry over the ring $\mathbf{C}[t]$.
It is well known that the variety $\mathrm{Spec}(\mathbf{C}[t])$ can be compactified to the projective line $\mathbf{P}_1$.
The field of fractions of $\mathbf{C}[t]$ is $\mathbf{C}(t)$ which is also the field of rational functions of $\mathbf{P}_1$.
To generalise a bit the situation, form now on we will fix a smooth projective curve B over $\mathbf{C}$ and we will denote by F the field $\mathbf{C}(B)$.
We will be interested in studying the Diophantine Geometry of *curves* over F.
Observe that the field F is not algebraically closed.

Let $f : X_F \to \mathrm{Spec}(F)$ be a variety defined over F. We would like to understand the set of F–rational points of X_F. We begin by give a precise definition of a rational point:

We start with some examples:

Example 1.1. Let $F = \mathbf{C}(t)$ and X_F be the line $\{X + tY + tZ = 0\} \subset \mathbf{P}^2$. Then the point $[0 : 1 : -1]$ is a F–rational point of X_F. But also $[-t; 1; 0]$ is a rational point of X_F.

In the example above we see that the two points on the variety X_F are of different nature: the first one has coordinates in $\mathbf{C}$ and the second had coordinates over the field F. Also notice that, in order to define the points and the curve, we used coordinates (thus an embedding inside a projective plane).

We would like to provide a geometric definition of rational point which is intrinsic, in particular it will depend on the coordinates or the particular embedding of the variety. .

We make another example:

Example 1.2. Let k be *any* field. Let X be the variety $\mathbf{A}^1_k := \mathrm{Spec}(k[t])$. A k rational point on X is simply an element $a \in k$. It correspond to the maximal ideal $(t - a) \subset k[t]$. It also corresponds to a k–morphism $k[t] \to k$. And it also corresponds to a k–morphism $a : \mathrm{Spec}(k) \to X$.

In general we see that the example above generalises to any affine variety: If A is a k algebra and $X = \mathrm{Spec}(A)$; Suppose that $X \subset \mathbf{A}^N_k$. A point $(a_1, \ldots, a_N)$ of $\mathbb{A}^N_k$ with coordinates in k which is contained in X corresponds to a maximal ideal $m_a \subset A$ such that $A/m_a \simeq k$.

We deduce the following important observation (which is essentially the statement of the classical Hilbert Nullstellensatz):

There is a bijection between k–rational points of X and morphisms of k-schemes $\mathrm{Spec}(k) \to X$.

From this observation, it is natural to give the following definition:

Definition 1.3. *Let F be a field and $f : X_F \to \mathrm{Spec}(F)$ be a F–variety. A F–rational point of X_F is a morphism of F–Schemes*

$$P : \mathrm{Spec}(F) \longrightarrow X_F.$$

The set of F–rational points of X_F is denoted $X_F(F)$.

Observe that, if L/F is a field extension and X_F is a variety defined over F, denoting X_L the L–variety $X_F \times_F \mathrm{Spec}(L)$, we have that

$$X_F(F) \subseteq X_L(L).$$

(prove this by exercise).

1.1. **Models of varieties.** Suppose again that $F = \mathbf{C}(t)$. Let X_F be an affine variety defined over K. By definition $X_F = \mathrm{Spec}(F[X_1, \ldots, X_n]/I)$, where $I = (G_1(X), \ldots, G_r(X))$ is the ideal specifying the equations which define X_F. Each $G_i(X)$ is a polynomial in $\mathbf{C}(t)[X_1, \ldots, X_n]$.

Suppose that $G_i(X) \in \mathbf{C}[t][X_1, \ldots, X_n]$.

In this case, we can associate to X_F the variety $\mathscr{X}$ *defined over* $\mathbf{C}$ given by $\mathscr{X} := \mathrm{Spec}(\mathbf{C}[t, X_1, \dots, X_n]/(G_1(t, X), \dots, G_r(t, X)))$. Roughly speaking, $\mathscr{X}$ have been builded up from X_F by considering the element $t \in K$ not as a constant but as a variable.

The natural morphism $\mathbf{C}[t] \to \mathbf{C}[t][X_1, \dots, X_n]$ give rise to a morphism

$$\mathscr{X} \to \mathbf{A}^1_{\mathbf{C}} \tag{1.2}$$

We also have a natural morphism (the generic point) $\eta : \mathrm{Spec}(F) \to \mathbf{A}^1_{\mathbf{C}}$ and a morphism of schemes $X_F \to \mathscr{X}$ given by the natural morphims $\mathbf{C}[t, X_1, \dots, X_n]/(G_1(t, X), \dots, G_r(t, X)) \to F[X_1, \dots, X_n]/I$. Moreover the diagram

$$\begin{array}{ccc} X_F & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \xrightarrow{\eta} & \mathbf{A}^1_{\mathbf{C}} \end{array} \tag{1.3}$$

is cartesian.

Let F be the field $\mathbf{C}(B)$ as above and let $X_F \to \mathrm{Spec}(F)$ be a F–variety. *A model* of X_F over B will be a $\mathbf{C}$–variety which generalise the construction above. We will see that the models are not unique and we will see the relations between them.

The field F is equipped with a map of k–schemes $\eta : \mathrm{Spec}(F) \to B$. The image is a point everywhere dense called *the generic point of* B.

Definition 1.4. *Let* $f : X_F \to \mathrm{Spec}(F)$ *be a variety. A* model *of* X_F *over* B *is a* $\mathbf{C}$–*variety* $\mathscr{X}$ *equipped with a* flat *surjective map of* $\mathbf{C}$ *varieties* $g : \mathscr{X} \to B$ *and such that* $X_F = \mathscr{X} \times_B \mathrm{Spec}(F)$*; in other words the following diagram is cartesian:*

$$\begin{array}{ccc} X_F & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \xrightarrow{\eta} & B. \end{array} \tag{1.4}$$

Example 1.5. Consider the curve X_F over field $F = \mathbf{C}(t)$ defined by the equation $y^2 = x^3 + t^2$. We may consider t as a variable and associate to X_F the *surface* $\mathscr{X}$ defined over $\mathbf{C}$ defined by the same equation (the variables will be x, y *and* t). The surface $\mathscr{X}$ has a k–morphism $f : \mathscr{X} \to \mathbb{A}^1_{\mathbf{C}}$; sending $(x; y; t)$ to t. Each time we fix a point $t_0 \in \mathbb{A}^1_{\mathbf{C}}$ we may look to the fibre of f over t_0; it will be the $\mathbf{C}$–curve $X_{t_0} := \{y^2 = x^3 + t_0^2\}$. Observe that the curve X_{t_0} is smooth if $t_0 \neq 0$ and singular if $t_0 = 0$. Observe also that the surface $\mathscr{X}$ is singular exactly where the curve X_0 is singular. If we blow up the singular point and then we blow up once again the singular point of the strict transform of $\mathscr{X}$ be obtain a smooth surface $\widetilde{\mathscr{X}}$. This is *another model of* X_F. Observe that this model has again a map over $\mathbb{A}^1_{\mathbf{C}}$ but this time the fibre over 0 is a reducible curve. On the other side, the fibre of $\widetilde{\mathscr{X}}$ over 0 remains singular.

1.2. **Main properties of models.**

(a) by definition, X_F is the fibre over the generic point of B. Since η is dense in B, we have that the image of X_F in $\mathscr{X}$ is dense.

(b) Let $F(X_B)$ be the fraction field of X_F, then $F(X_F) = \mathbf{C}(\mathscr{X})$ (prove it by exercise).

(c) Let $p \in B$ be a closed point and $Y \subset \mathscr{X}$ be a subvariety contained in the fibre over p; Let $\widetilde{\mathscr{X}} \to \mathscr{X}$ be the blow up of Y; then $\widetilde{\mathscr{X}}$ is another model of X_F.

(d) More generally you can prove the following fact:

Proposition 1.6. *Let $\mathscr{X}$ be a $\mathbf{C}$–variety with an isomorphism of fields $F(X_F) \simeq \mathbf{C}(\mathscr{X})$ then, there is a variety $\widetilde{\mathscr{X}}$ birational to $\mathscr{X}$ with a flat surjective map $\widetilde{\mathscr{X}} \to B$ which is a model of X_F.*

We can also clarify the relationship within two arbitrary models of the same projective variety:

Proposition 1.7. *Suppose that $\mathscr{X}_1$ and $\mathscr{X}_2$ are two projective models of the same variety X_F. Then, there exists a third model $\mathscr{X}_3$ of X_F over B with a commutative diagram*

(1.5)
$$\begin{array}{ccc} & \mathscr{X}_3 & \\ \swarrow & & \searrow \\ \mathscr{X}_1 & \dashrightarrow & \mathscr{X}_2 \end{array}$$

where the dotted arrow is a birational map.

(e)A natural model for the projective space $\mathbf{P}^n_F$ is the product $B \times_{\mathbf{C}} \mathbf{P}^n_{\mathbf{C}}$. More generally, if X_0 is a variety defined over $\mathbf{C}$, then a "natural model for the F–variety $X_0 \times_{\mathbf{C}} \mathrm{Spec}(F)$ is the variety $X_0 \times B$.

(f) Suppose that X_F is projective, an easy (but not always good) way to construct a model of X_F is the following:

Embed $X_F \hookrightarrow \mathbf{P}^N_F$. Consider the model of $\mathbf{P}^N_F$ constructed in (e) and take the Zariski closure of X_F inside it. The Zariski closure $\mathscr{X} \hookrightarrow B \times_{\mathbf{C}} \mathbf{P}^N_k$ is then a model of X_F.

(g) Other natural models of $\mathbf{P}^N_F$ are constructed as follows: Let E be a vector bundle of rank $N+1$ over B, then $\mathbf{P}(E) \to B$ is a smooth projective model of $\mathbf{P}^N_F$; of course we can apply (f) to this situation and construct in this way other models of projective X_F.

(h) Suppose that R is a $\mathbf{C}$–scheme such that $\mathbf{C}(R) = F$. Typical examples R are affine open sets of B or the spectra of local rings of closed points of B. As before we have a natural map $\eta : \mathrm{Spec}(F) \to R$. We can generalise the notion of model to R: It will be a faithfully flat R–scheme $\mathscr{X}_R \to R$ such that $X_F \simeq \mathscr{X}_R \times_R \mathrm{Spec}(F)$. In this case we will say that $\mathscr{X}_R$ is a model of $\mathscr{X}_F$ over R.

(i) Suppose that $\mathscr{X}$ is a model of X_F over B. Let $p_0 : \mathrm{Spec}(\mathbf{C}) \to B$ be a closed point. Then we may consider the $\mathbf{C}$–variety $X_{p_0} := \mathscr{X} \times_B p_0$. This construction is the formal version of the argument of "specialising the parameters" used in the example 1.5.

(j) Suppose that $h : B' \to B$ is a finite covering of curves. Let F' be the field of functions of B' (it is a finite extension of F). Let $\mathscr{X} \to B$ be a model of X_F. Let $X_{F'}$ be the F'–variety

$X_F \times_F \mathrm{Spec}(F')$. Then $\mathscr{X}' =: \mathscr{X} \times_B B'$ is a model of $X_{F'}$ over F'. Warning: even if $\mathscr{X}$ is a $\mathbf{C}$–smooth variety, in general $\mathscr{X}'$ will not be a smooth variety.

(k) By the theorem of resolution of singularities (which holds over $\mathbf{C}$), every smooth projective variety X_F over F has a model which is a smooth projective $\mathbf{C}$–variety. In this case we will call such a model *a projective regular model of X_F over B*.

(l) Suppose that X_F is smooth. From now on we will always suppose that the models of it are normal. Observe that every model is dominated by a normal model.

1.3. **Models and rational points.** We would like to understand the relations between models and rational points. If the variety X_F is projective then the F–rational points of X_F may be described geometrically in terms of models and morphisms of $\mathbf{C}$–varieties.

Let $f : X_F \to \mathrm{Spec}(F)$ be a projective variety over F.

Let's analyze first a simplified case.

Suppose that there exists a variety X_0 defined over $\mathbf{C}$ such that $X_F = X_0 \times_{\mathbf{C}} \mathrm{Spec}(F)$. Thus a model of X_F is $X_0 \times B$. Each time we have a $\mathbf{C}$–morphism of curves $P : B \to X_0$, we can look to its graph $\Gamma_P : B \to X_0 \times_k B$.

The restriction of Γ_P to the generic point $\mathrm{Spec}(F)$ of B give rise to a point $P_F \in X_F(F)$.

Thus we get an inclusion $Hom_{\mathbf{C}}(B; X_0) \subseteq X_F(F)$. Since X_0 is projective, this inclusion is indeed an equality:

Proposition 1.8. *If X_0 is projective then $Hom_{\mathbf{C}}(B; X_0) = X_F(F)$.*

Proof. We need to prove that any point $p \in X_F(F)$ comes from a morphism $P : B \to X_0$. We first do the case when $X_0 = \mathbf{P}^N$. Fix coordinates on $\mathbf{P}^N_k$. A rational point $p \in \mathbf{P}^N(F)$ corresponds to $N + 1$ rational functions $[f_0; \ldots ; f_N]$ up to a non trivial scalar factor. By definition $[f_0; \ldots ; f_N]$ defines a morphism from an affine open set of B to $\mathbf{P}^N$. Since B is smooth, this extends to a morphism from B to $\mathbf{P}^N$.

Exercise 1.9. In the proof of the proposition above we used the following fact: Let B be a smooth projective curve and U be a Zariski open set of it. Let $f : U \to \mathbf{P}^N$ be a morphism. Then f extends to a morphism $f' : B \to \mathbf{P}^N$. Prove it.

We have natural maps $X_F \to X_0 \times B \to \mathbf{P}^N \times B$ and the closure of the image of X_F is $X_0 \times B$. Every point $p : \mathrm{Spec}(F) \to X_F$ extends to a morphism morphism $P : B \to \mathbf{P}^N \times B$ whose projection on B is the identity (because of the first part of the proof); its image is contained in $X_0 \times B$ because $X_0 \times B$ is the closure of X_F in $\mathbf{P}^N \times B$. □

It is important to notice that the same proof applies in the general case:

Theorem 1.10. *Let X_F be a projective variety over F and $f : \mathscr{X} \to B$ be a projective model of it. Then there is a bijection:*

$$\{\text{Points } p \in X_F(F)\} \longleftrightarrow \{\mathbf{C} - \text{morphisms } P : B \to \mathscr{X} \text{ s.t. } f \circ P = Id\}.$$

The proof in the general case is a consequence of the case of $\mathbf{P}^N$ and the Lemma below (and then left as exercice).

Lemma 1.11. *Let $f : \mathscr{X} \to B$ be a model of X_F. Suppose that $\mathscr{X}$ is a smooth $\mathbf{C}$–variety. Then the morphism $f : \mathscr{X} \to B$ can be factorized as $f = p \circ i$ with $i : \mathscr{X} \to \mathbf{P}^N \times B$ a closed immersion and $p : \mathbf{P}^N \times B \to B$ is the second projection.*

Proof. Since $\mathscr{X}$ is projective over k; we can embed it inside some projective space $\iota\mathscr{X} \to \mathbf{P}^n$. The map $i := (\iota; f) : \mathscr{X} \to \mathbf{P}^N \times B$ has the searched properties. □

The lemma above is useful because it will reduce the verification of many properties to the "easy" case of $\mathbf{P}^N_F$ with the corresponding trivial model. Observe that $\mathscr{X}$ is a closed subvariety of $\mathbf{P}^N \times B$.

The theorem above give a geometric interpretation of F rational points of a projective variety: they correspond to $\mathbf{C}$ morphisms of the projective curve B to a model of the variety which composed with the structural morphism are the identity.

Remark 1.12. Observe that a rational point is the generic fibre of a section.

The following example shows that the hypothesis of projectiveness is essential.

Example 1.13. Let $F := \mathbf{C}(t)$. The corresponding curve is $\mathbf{P}^1_k$. Consider the variety $X_F := \mathbf{A}^1_F$. Consider the point $p \in X_F(F)$ with coordinate t. This point *do not extend to a morphism* $P : \mathbf{P}^1_k \to \mathbf{P}^1_{\mathbf{C}} \times \mathbf{A}^1_k$. Indeed such a point will give a non trivial morphism from $\mathbf{P}^1_k$ to $\mathbb{A}^1_k$ and this is impossible.

1.4. **Models of line bundles.** It is important to observe that, if $f : \mathscr{X} \to B$ is a model of X_F, $p \in X_F(F)$ is a rational point and the we may see the section $P : B \to \mathscr{X}$ corresponding to p as *a model of the morphism* $p : \mathrm{Spec}(F) \to X_F$. In general, we can have models of many "objects" defined over X_F (cycles, divisors, sheaves, vector bundles...): these will be objects of the same nature over a model cX of X_F whose restriction to the generic fibre is the object itself.

In particular we can have models of line bundles:

Let L_F be a line bundle over X_F. Fix a model $f : \mathscr{X} \to B$ of X_F over B. In the sequel, we will denote by $\eta : X_F \to \mathscr{X}$ the natural inclusion.

Definition 1.14. *A model of L_F is a line bundle $\mathscr{L}$ over $\mathscr{X}$ such that, if $\eta : X_F \to \mathscr{X}$ is the natural inclusion, then $\eta^*(\mathscr{L}) \simeq L_F$.*

(a) Suppose that X_F is a curve. The degree of the restriction of the line bundle $\mathscr{L}$ to the fibre over a closed point of B do not depend on the point. This number coincide with the degree of the line bundle L_F.

(b) If $\mathscr{L}$ is a line bundle on $\mathscr{X}$ whose restriction to the generic fibre is isomorphic to L_F, then $\mathscr{L}$ will be a model of L_F.

(c) If X_F is $\mathbf{P}^N_F$ then a natural model of $\mathscr{O}(1)$ over the model $\mathbf{P}^N_k \times B$ is $p_1^*(\mathscr{O}(1))$ where $p_1 : \mathbf{P}^N_k \times B \to \mathbf{P}^N_k$ is the the first projection. We will denote this model, again by $\mathscr{O}(1)$.

It is very important to observe that models of line bundles always exist over suitable models of X_F:

Theorem 1.15. *Let X_F be a smooth projective variety over F and L_F be a line bundle over it. Let $f : \mathscr{X} \to B$ be a projective model of X_F. Then there is a blow up $\widetilde{\mathscr{X}}$ of $\mathscr{X}$ and a model $\mathscr{L}$ of L_F over $\widetilde{\mathscr{X}}$.*

Proof. Every line bundle L_F on X_F may be written as $L_F = M \otimes N^{\otimes -1}$ where M is generated by global sections and N is very ample (thus generated by global sections). Consequently it suffices to prove the theorem when L_F is generated by global sections.

Since L_F is generated by global sections, it defines a morphism $g_{L_F} : X_F \to \mathbf{P}^N_F$ for a suitable N; moreover $g^*_{L_F}(\mathscr{O}(1)) = L_F$. This morphism extends to a rational map $g : \mathscr{X} \dashrightarrow \mathbf{P}^N \times B$. Consequently there is a commutative diagram

$$
\begin{array}{ccc}
 & \mathscr{X}_1 & \\
 \swarrow & & \searrow \\
\mathscr{X} & \dashrightarrow & \mathbf{P}^N \times B
\end{array}
\tag{1.6}
$$

where the continuous arrows are morphisms. The variety $\mathscr{X}_1$ is birational to $\mathscr{X}$ and it is model of X_F over B. The line bundle $g^*(\mathscr{O}(1))$ is a model of L_F. □

A similar argument gives:

Proposition 1.16. *Suppose that L_F is an ample line bundle over a smooth projective variety X_F. Then one can find a regular projective model $f : \mathscr{X} \to B$ of X_F , a positive integer n and an ample line bundle $\mathscr{L}_1$ of $L_F^{\otimes n}$ over it.*

In order to prove this, we recall the following general fact:

d) Suppose that Z is a projective variety and L is an ample line bundle over it. Let $p : \widetilde{Z} \to Z$ be a blow up of Z with exceptional divisor E. Then there exists a positive integer n such that the line bundle $p^*(L^{\otimes n})(-E)$ is ample on $\widetilde{Z}$.

Proof. There exists an integer m for which the linear system $H^0(X_F, L_F^M)$ gives an embedding of X_F in some projective space $\mathbf{P}^r$. The Zariski closure $\overline{X}$ of X_F in $B \times \mathbf{P}^r$ is a model of X_F equipped with a line bundle $\overline{L}$ which is ample. Let $p : \mathscr{X} \to \overline{X}$ be a resolution of singularities which is is obtained by successive blow up and let E be the exceptional divisor. The line bundle $p^*(\overline{L}^{mn})(-E)$ will be an ample model of L_F^{mn}. □

2. Lecture Two: Curves of genus zero over function fields

It is well known (Lüroth theorem) that every curve of genus zero over a algebraically closed field is isomorphic to the projective line $\mathbf{P}^1$. If the base field is not algebraically closed it is possible to have curves of genus zero which are not isomorphic to $\mathbf{P}^1$.

Example 2.1. Let $X = \{X_0^2 + X_1^2 + X_2^2 = 0\} \subset \mathbf{P}^2_{\mathbf{Q}}$. The curve $X_{\mathbf{Q}}$ is of genus zero (it is a conic) but it is not isomorphic to the projective line over $\mathbf{Q}$: indeed $X(\mathbf{Q}) = \emptyset$ and any isomorphism defined over a field send rational points to rational points.

Let F be a function field in one variable over the complex field as before and X_F be a curve of genus zero over it. In this lecture we will prove that there is an isomorphism $X_K \simeq \mathbf{P}^1_F$ *defined over* F.

The proof we propose uses the complex topology. It is possible to find other proofs which hold over any function field over an algebraically closed field.

The fact that every curve of genus zero over a function field of one variable F is isomorphic to $\mathbf{P}^1_F$ will be consequence of the following two theorems:

Theorem 2.2. *Let F be a field and X_F be a smooth projective curve of genus zero. Suppose that $X_F(F)$ is not empty, then there exists an isomorphism $X_F \simeq \mathbf{P}^1_F$ defined over F.*

Proof. Let $\overline{F}$ be the algebraic closure of F. Since X_F is of genus zero, the $\overline{F}$ curve $\overline{X};= X_F \otimes_F \overline{F}$ is isomorphic to $\mathbf{P}^1_{\overline{F}}$. If K_{X_F} is the canonical line bundle of X_F then we have that $-K_X$ is very ample. Indeed the restriction of $-K_X$ to $\overline{X}$ is isomorphic to $\mathscr{O}_{\mathbf{P}^1}(2)$.

The linear system $H^0(X_F, -K_{X_K})$ embeds X_K as a conic (smooth irreducible curve of degree two) in $\mathbf{P}^2_F$. Thus every curve of genus zero over a field can be realized as a conic in the projective line.

Let $C \subset \mathbf{P}^2_F$ be a conic. Suppose that $C(F)$ is non empty. Let $P \in C(F)$; every line r in $\mathbf{P}^2_F$ defined over F and passing through P intersects the conic in P and another point Q_r.

The point Q_r is a F–rational point of C. Indeed every line defined over F intersects C in two points which are conjugate. If one of the two points is F–rational, the other must be rational too.

The linear system of the lines of $\mathbf{P}^2_F$ passing through P is isomorphic to $\mathbf{P}^1_F$ and the map $r \to Q_r$ defines a rational map between $\mathbf{P}^1_F$ and C which is generically a bijection. Since C is smooth, this map is an isomorphism. □

Exercise 2.3. Let $F = \mathbf{C}((t))$ and $C = \{X^2 + tY^2 - t^2Z^2 = 0\} \subset \mathbf{P}^2_F$. Find an explicit isomorphism within C and $\mathbf{P}^1_F$ defined over F.

The second Theorem is more difficult:

Theorem 2.4. *Let F be the function field of an algebraic curve B defined over $\mathbf{C}$ and X_F be a smooth projective curve of genus zero. Then $X_F(F) \neq \emptyset$.*

We will propose a proof of this theorem which holds over the complex numbers: we suppose that the curve B is a curve defined over $\mathbf{C}$. One can give a proof which holds over a general algebraic closed field. The statement do not hold for arbitrary field.

In order to prove the theorem we need to recall some general properties of intersection theory on surfaces:

– Let S be a smooth variety over the complex numbers. Let $\mathscr{O}_S$ be the structural sheaf of S and $\mathscr{O}^*_S$ be the sheaf of invertible holomorphic functions on S. Let $\underline{\mathbf{Z}}_S$ be the sheaf of locally constant continuous functions of S with values in $\mathbf{Z}$, then we have an exact sequence of sheafs

$$0 \to \underline{\mathbf{Z}}_S \longrightarrow \mathscr{O}_S \stackrel{\exp(2\pi i(\cdot))}{\longrightarrow} \mathscr{O}^*_S \to 0. \tag{2.1}$$

This give rise to an exact sequence of groups

$$(2.2)\qquad \dots \to H^1(S,\mathbf{Z}) \longrightarrow H^1(S,\mathcal{O}_S) \longrightarrow H^1(S,\mathcal{O}_S^*) \longrightarrow H^2(S,\mathbf{Z}) \longrightarrow H^2(S,\emptyset_S) \to \dots .$$

– The group $H^1(S,\mathcal{O}_S^*)$ is canonically isomorphic to the Picard group $Pic(S)$ of isomorphism classes of line bundles on S. Thus the exact sequence above give rise to a morphism of groups

$$\delta : Pic(S) \longrightarrow H^2(S;\mathbf{Z}).$$

Now we list some properties of surfaces which will be used in the proof:

a) (Poincaré duality) In this case the cup product $H^2(S,\mathbf{Z})\otimes_{\mathbf{Z}} H^2(S,\mathbf{Z}) \to H^4(S,\mathbf{Z}) \simeq \mathbf{Z}$ is a perfect pairing.

b) (Serre Duality) Let Ω^1_S be the locally free sheaf of differentials of S and $K_S := \Lambda^2(\Omega^1_S)$. Then K_S is a line bundle on S called the *canonical class* of S. If L is a line bundle on S then we have a canonical isomorphism

$$(2.3)\qquad H^2(S,L) \simeq H^0(S,K_S\otimes L^\vee)^\vee$$

where $(\cdot)^\vee$ is the dual of $(\cdot)$.

c) (Riemann Roch) If L is a line bundle on S, then the Euler characteristic of L is the integer $\chi(L) := \sum_{i=0}^2(-1)^i\dim(H^i(S,L))$. Then

$$\lim_{n\to\infty}\frac{\chi(L^{\otimes n})}{n^2} = \frac{(L;L)}{2}.$$

d) Let D_1 and D_2 be divisors in S. Then $(D_1;D_2) = (\delta(\mathcal{O}_S(D_1);\delta(\mathcal{O}_S(D_2))$. Thus the algebraic intersection product coincides with the topological cup product.

e) (Adjunction formula) Suppose that D is a smooth curve in S then

$$(K_S;D) + (D;D) = 2g(D) - 2.$$

In particular, since every class of line bundle is difference of two very ample line bundles and a very ample line bundle is numerically equivalent to a smooth curve on S we find:

f) If A is a class of divisors on S then $(K_S;A) + (A:A) \equiv 0 \; mod(2)$.

g) A divisor H on S is said to be *nef* (numerically effective) if, for every effective divisor D on S we have $(H,D) \geq 0$. If H is nef and A is an ample divisor on S then for every positive integer n the divisor $nH + A$ is ample (this is a consequence of the Nakai Moishezon ampleness criterion).

h)

Proposition 2.5. *If H is a nef divisor and D is a divisor such that $(H,D) > 0$ and $(D,D) > 0$ then, for $n \gg 0$ we have $H^0(S,\mathcal{O}(nD)) \neq 0$.*

Proof. Since $(D;D) > 0$, for $n \gg 0$ we have, by Riemann Roch, $\chi(\mathcal{O}(nD)) > 0$. Suppose that $H^2(S,\mathcal{O}(nD)) \neq \{0\}$ then by Serre duality, we can find a non zero effective divisor E on S linearly equivalent to $K_S\otimes\mathcal{O}(-nD)$. But, for n sufficiently big, $(K_S\otimes\mathcal{O}(-nD),H) < 0$ and this is not possible. Thus $H^2(S,\mathcal{O}(nD)) = \{0\}$ and $H^0(S,nD) \neq \{0\}$. □

We can now start the proof of Theorem 2.4.

Proof. (Of 2.4) We fix a model $f : \mathscr{X} \to B$ of X_F where $\mathscr{X}$ is a smooth projective surface and f is flat. Let $b \in B(\mathbf{C})$ be a sufficiently general point and F_b the fibre $f^*(b)$. Then:

– F_b is a smooth projective curve in $\mathscr{X}$ which is of genus zero (because X_F is) and it is nef as a divisor (prove it by exercise). SInce two fibers are disjoint and numerically equivalent, we have that $(F_b, F_b) = 0$.

– Let $K_{\mathscr{X}}$ be the canonical class of $\mathscr{X}$. By adjunction formula, we have $(K_{\mathscr{X}}; F_b) = -2$.

– We have that $H^2(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) = \{0\}$: Indeed suppose that $H^2(\mathscr{X}; \mathscr{O}_{\mathscr{X}}) \neq \{0\}$, then, by Serre duality we have $H^0(\mathscr{X}, K_{\mathscr{X}}) \neq \{0\}$. But $(K_{\mathscr{X}}; F_b) = -2$ and F_b is nef. Contradiction.

– The remark above implies that the map $\delta : Pic(\mathscr{X}) \to H^2(\mathscr{X}, \mathbf{Z})$ is surjective. Consequently every class in $H^2(\mathscr{X}, \mathbf{Z})$ is represented by a line bundle (or a divisor).

Consider the linear map

$$(\cdot; \delta(\mathscr{O}(F_b)) : H^2(\mathscr{X}; \mathbf{Z}) \longrightarrow \mathbf{Z} \qquad (2.4)$$

$$a \longrightarrow (a; \delta(\mathscr{O}(F_b)).$$

The image of the map will be a subgroup of $\mathbf{Z}$ so it will be of the form $d\mathbf{Z}$.

Let's prove that $d = 1$. The linear map

$$\alpha_b := \tfrac{1}{d}(\cdot; \delta(\mathscr{O}(F_b)) : H^2(\mathscr{X}; \mathbf{Z}) \longrightarrow \mathbf{Z} \qquad (2.5)$$

$$a \longrightarrow \tfrac{1}{d}(a; \delta(\mathscr{O}(F_b)).$$

is well defined and, by Poincaré duality, there exists a class $A \in H^2(\mathscr{X}, \mathbf{Z})$ such that $(\cdot; A) = \alpha_b(\cdot)$. The class A is the image, via δ of a divisor , which we will denote again by A. Since $d \cdot (A; F_b) = d \cdot \alpha_b(F_b) = (F_b, F_b) = 0$ and $d \cdot (A; A) = d \cdot \alpha_b(A) = (A, F_b)$ we obtain that $(A, A) = 0$. Moreover, by the adjunction formula we have

$$-2 = (K_X; F_b) = d(K_{\mathscr{X}}; A) \qquad (2.6)$$

But, as remarked above, $(K_{\mathscr{X}}, A) + (A, A) \equiv 0 \; mod(2)$, thus the only possibility is $d = 1$.

We proved then that there exists a divisor B_1 on $\mathscr{X}$ such that $(B_1; F_b) = 1$.

We claim that $B_1 + nF_b$ is effective for $n \gg 0$. Indeed $(B_1 + nF_B; F_b) = 1 > 0$ and $(B_1 + nF_b; B_1 + nF_b) = (B_1 : B_1) + 2n > 0$ as soon as $n \gg 0$. By Riemann Roch, $\chi(\mathscr{O}(B_1 + nF_b)) > 0$ for $n \gg 0$. Suppose that $H^2(\mathscr{X}, \mathscr{O}(B_1 + nF)) \neq \{0\}$ then by Serre duality, we can find a non zero effective divisor E on S linearly equivalent to $K_{\mathscr{X}} \otimes \mathscr{O}(-B_1 - nF_b)$. But, $(K_{\mathscr{X}} \otimes \mathscr{O}(-B_1 - nF_b), F_b) = -3 < 0$ and this is not possible because F_b is nef. Thus $H^2(\mathscr{X}, \mathscr{O}(nD)) = \{0\}$ and $H^0(\mathscr{X}, B_1 + nF_b) \neq \{0\}$.

We proved that there exists *an effective divisor* B_2 such that $(B_2; F_b) = 1$. Write $B_2 = B_h + V$ where B_h dominates B and $f(V)$ is a proper closed set of B. Let B_3 be an irreducible component of B_h (indeed you can prove, by exercice, that B_h must be irreducible). The morphism $f_{B_3} : B_3 \to B$ is finite and of degree one. Since B is smooth, it is an isomorphism. Thus B_3 is a section of f. The restriction of f to the generic fibre is a rational point of X_F. □

3. Lecture Three: Isotriviality

Let F be a function field as in the lectures before and B the smooth projective curve associated to it. Within all the curves X_F defined over F there are the curves which are defined over $\mathbf{C}$. Indeed, if X_0 is a curve defined over $\mathbf{C}$, then the curve $X_0 \times_{\mathbf{C}} \mathrm{Spec}(F)$ is a curve defined over F.

Over F there can be also curves which are obtained in this way only after a finite extension of F

Example 3.1. Let $F = \mathbf{C}(t)$. It is the field of fractions of $\mathbf{P}^1$. Consider the Curve $C := \{Y^2 = X^3 + t\}$. A priori it seems that C is a curve defined over F and which is *not* defined over $\mathbf{C}$. But consider the extension $L := F[t^{1/2}, t^{1/3}]$; over L the change of variables $X = t^{1/3}X_1$ and $Y = t^{1/2}Y_1$ give rise to an isomorphism of C with the curve $Y_1^2 = X_1^3 + 1$ which is defined over $\mathbf{C}$.

Curves of this kind play a special role in Diophantine Geometry thus we introduce the following definition:

Definition 3.2. *Let X_F be a curve defined over F. We will say that X_F is* isotrivial *if there exists a curve X_0 defined over $\mathbf{C}$ and an isomorphism*

$$X_F \times_F \overline{F} \simeq X_0 \times_{\mathbf{C}} \overline{F} \tag{3.1}$$

where $\overline{F}$ is the algebraic closure of F.

Of course, if such an isomorphism exists, it is defined over a finite extension of F.

If X_F is the curve $X_0 \times_{\mathbf{C}} \mathrm{Spec}(F)$ described before, a natural model of X_F over B is the surface $\mathscr{X} = X_0 \times B$ with the natural projection $p : X_0 \times B \to B$. Observe that for every $b \in B$, the fibre $\mathscr{X}_b$ of p over p is the curve X_0. *In particular all the fibres of the projection p are isomorphic.*

This property is specific to isotrivial curves.

Remark that in the lecture before we proved that every curve of genus zero is isotrivial.

Theorem 3.3. *Let X_F be a smooth projective curve of genus at least two defined over F. Then X_F is isotrivial if and only if for every projective model $p : \mathscr{X} \to B$ of X_F there exists a non empty open set $U_{\mathscr{X}} \subset B$ such that, for every couple of closed points b_1 and b_2 of $U_{\mathscr{X}}$ the fibre $\mathscr{X}_{b_1}$ is isomorphic to the fibre $\mathscr{X}_{b_2}$.*

A similar statement holds for curves of genus one but one has to suppose that they have a rational point and one has to define isotriviality for curves of genus one with a markerd point.

Before we start the proof we recall the following fact from algebraic geometry:

a) Let $\mathscr{X} \to B$ and $\mathscr{Y} \to B$ be two families of projective curves of genus at least two. Suppose that their generic fibre is smooth. Then there exists a scheme $h : Isom_B(\mathscr{X}, \mathscr{Y}) \to B$ with the following properties:

i) $Isom_B(\mathscr{X}, \mathscr{Y})$ represents the functor of B–isomorphims from $\mathscr{X}$ to $\mathscr{Y}$: for every $T \to B$, an element of $Isom_B(\mathscr{X}, \mathscr{Y})(T)$ is an $T-$ isomorphism from $\mathscr{X} \times_B T$ to $\mathscr{Y} \times_B T$.

ii) $Isom_B(\mathscr{X}, \mathscr{Y})$ is finite over B.

For more details about the scheme $Isom_B(\mathscr{X}; \mathscr{Y})$, cf. for instance, [8] and [5].

We can now start the proof of Theorem 3.3:

Proof. Let X_F be a F–curve. For the time being we will say that a projective model $\mathscr{X} \to B$ of X_F has the "isotrivial property" if there exists a open set $U_{\mathscr{X}} \subset B$ such that, for every couple of closed points b_1 and b_2 of $U_{\mathscr{X}}$ the fibre $\mathscr{X}_{b_1}$ is isomorphic to the fibre $\mathscr{X}_{b_2}$.

We claim that X_F has a model which have the isotrivial property, then the isotrivial property holds for every projective model of X_F.

Indeed two models are birationally equivalent and they are dominated by a third one. Thus it suffices to prove the claim when one model dominates the other. In this case, the exceptional divisors do not dominate B thus the fibers of one model coincide with the fibers of the other over a open set of B.

Moreover, if there is a finite extension L/K such that the curve $X_L := X_F \times_F L$ has a model with the isotrivial property, then every model of X_F has the isotrivial property (details as exercise).

This implies that if X_F is isotrivial, then every model of it has the isotrivial property.

Conversely, suppose that X_F has a model $\mathscr{X} \to B$ which has the isotrivial property. There exists then a open set $U_{\mathscr{X}} \subset B$ such that, for every $b \in U_{\mathscr{X}}$ the fibre $\mathscr{X}_b$ is isomorphic to a fixed curve X_0 (defined over $\mathbf{C}$).

Consider the B scheme $Isom_B(\mathscr{X}, B \times X_0)$. It is finite over B (by property (b) above) and dominant (by hypothesis). Thus there is a finite covering $B' \to B$ with morphism $B' \to Isom_B(\mathscr{X}, B \times X_0)$. If we denote by L the field $\mathbf{C}(B')$, we then have that the curve X_L is isomorphic to $X_0 \times_{\mathbf{C}} L$. □

We will now prove an important criterion of isotriviality.

Theorem 3.4. *Let X_F be a smooth projective curve of genus at least two over F. Suppose that there exists an isotrivial curve Y_F and a finite morphism $f : Y_F \to X_F$. Then X_F is isotrivial.*

A similar statement hold for curves of genus one but one needs to require the presence of a rational point.

The theorem will be consequence of a classical Theorem by De Franchis:

Theorem 3.5. *(De Franchis) Let Y be a curve of genus $g \geq 2$. Then there exist only finitely many curves X with genus at least two and a non trivial morphism $h : Y \to X$.*

Let's see first how this implies Theorem 3.4:

Proof. (Of Theorem 3.4) We can make a field extension an suppose that Y_F is isomorphic to a curve Y_0 which is defined over $\mathbf{C}$.

Let $\mathscr{X} \to B$ be a model of X_F.

The finite morphism $f : Y_F \to X_F$ extends to a morphism from a blow up of $B \times Y_0$ to $\mathscr{X}$. Moreover the exceptional divisors of the morphisms do not dominate B.

This means that we can find an open set $U \subset B$ such that for every $b \in U$ we have a finite morphism $Y_0 \to \mathscr{X}_b$.

By De Franchis Theorem, this implies that the $\mathscr{X}_b$'s belong to a finite list. Thus there exists a smooth projective curve X_0 for with there are infinitely many $b \in U$ such that $\mathscr{X}_b \simeq X_0$.

We consider again the scheme $Isom_B(\mathscr{X}, X_0 \times B)$. It is finite and dominant over B (the image contains at least all the b's as above. Thus we conclude as in the proof of Theorem 3.3. □

We now come to the proof of the De Franchis's Theorem:

Proof. We first remark the following facts:

– By the Hurwitz formula, the genus of the possible X is bounded by the genus of Y.

– Moreover the degree of the possible h's is also bounded by a constant which depends only on the genus of Y.

– Consequently we may suppose that the genus of the X's and the degree of h are fixed.

Given a curve X with a morphism $h : Y \to X$, we can associate to it the "equivalence relation" R_h defined as follows:

(3.2)
$$\begin{array}{ccc} R_h & \longrightarrow & \Delta_X \\ \downarrow & & \downarrow \\ Y \times Y & \xrightarrow{(h,h)} & X \times X. \end{array}$$

The diagram being cartesian and Δ is the diagonal. Observe that $R_h = Y \times_X Y$. Moreover, as a divisor, $R_h = (h,h)^*(\Delta_X)$.

We claim that if $h : Y \to X$ and $k : Y \to Z$ are two finite maps as above, then $R_h = R_k$ if and only if $X \simeq Z$ and $h = k$ (up to isomorphism of X).

Consider the map $(h,k) : Y \times Y \to X \times Z$. Let $T := (h;k)(R_h)$. Since (h,k) is proper and finite, T is a divisor. It suffices to prove that the natural projection $T \to X$ is an isomorphism (by symmetry, T will be isomorphic to Z etc.)

Since X is smooth, it suffices to prove that $T \to X$ is of degree one.

Over an open set of X (and of Y), $R_h = \{(a,b) \in Y \times Y \ / \ h(a) = h(b)\}$; and similarly for R_k.

Thus, the fact that $R_h = R_k$ means that there exist open sets U_X of X and V_Z of Z such that, for every $x \in U_X$ there exists *a unique* $z \in U_Z$ such that $h^{-1}(x) = k^{-1}(z)$.

This implies that the restriction of T to $U_X \times V_Z$ is the set (x,y) where $h^{-1}(x) = k^{-1}(y)$. Consequently $T|_{U_X} \to U_X$ is a bijection thus the map $T \to X$ is of degree one as claimed.

In order to conclude, we need to prove that there are only finitely many such R_h.

We will prove the following two facts:

1) As a divisor, $R_h = \sum_{i=1}^{a} n_i R_h^i$ where each of the R_h^i dominates Δ_X, then there is a constant c_1, depending only on the genus of Y, such that, the multiplicity n_i of R_h^i in R_h and the number a of the components of R_h are both bounded by and c_1.

2) Given two morphisms $h : Y \to X_1$ and $k : Y \to X_2$ (with X_i of genus at least two) with associated divisors $R_i^h = \sum_{i=1}^{a} R_h^i$ and $R_k^j = \sum_{j=1}^{b} R_k^j$ respectively, then R_h^i is numerically equivalent to R_k^j if and only if $R_i^h = R_k^j$.

3) There exists an ample line bundle H on $Y \times Y$ and a constant c which depends only on the genus of Y, such that, for every morphism $h : Y \to X$ (with X of genus at least two) and integer i we have $(R_h^i; H) \leq c$.

To conclude we wii also need the following general fact:

Lemma 3.6. *If H is an ample class on a smooth projective surface X, and T is a positive constant, then the set A_T of numerical classes of effective divisors D on X such that $(D, H) \leq T$ is finite*

Proof. Let $H = H_1, \ldots, H_r$ be a basis of $Num(X)_{\mathbf{R}}$ made by ample divisors. Since $rH_1 - H_i$ is ample for r sufficiently big, we have that we may find a constant T_1 such that if $D \in A_T$ then $(D, H_i) \leq T_1$. The intersection product define an isomorphism between the vector space $Num(X)_{\mathbf{R}}$ and its dual. Let $\varphi_{H_1}, \ldots, \varphi_{H_r}$ be the dual basis of the basis above. An element $D \in A_T$ can be then written as $D = \sum a_i \varphi_{H_i}$ with $0 < a_i \leq T_1$. Since $Num(X)$ is discrete, the claim follows. □

Let's see how points (1)–(3) and the Lemma above imply the Theorem: Let $h : Y \to X$ be a morphism as above and $R_h = \sum_{i=1}^{a} R_h^i$ the associated divisor.

In order to conclude the proof, by point (1) it suffices to show that the R_H^i belong to a finite list which depends only on Y. By point (2) in order to show this, it suffices to show that the numerical class of the possible R_h^i belong to a finite list. Point (3) and Lemma 3.6 imply that the R_H^i belong to a finite list. Thus the conclusion of the Theorem Follows.

Let's show now points (1)– (3).

Denote by $f : Y \times Y \to X \times X$ the morphism (h, h). It is a finite morphism of degree $d := deg(h)^2$. Thus its degree is bounded only in terms of the genus of Y. In particular it does not contract any divisor of $Y \times Y$.

For every irreducible component R_h^i, the morphism $f_{R_H^i} : R_h^i \to \Delta_X = X$ is finite and its degree d_i is less or equal to d. Thus $f_*(R_H^i) = d_i \cdot \Delta_X$

Consequently we have that

$$d \cdot \Delta_X = f_*(f^*(\Delta_X)) = f_*(R_h) = \sum_{i=1}^{a} n_i \cdot (f_*(R_H^i)) = (\sum_{i=1}^{a} n_i \cdot d_i)\Delta_X. \tag{3.3}$$

As a consequence we find that the number a of irreducible components of R_h and each of the n_i are bounded by d, thus by a constant depending only on the genus of Y. Which proves (1).

Remark 3.7. By restriction to the étale part of the morphism h one can actually prove that $n_i = 1$ for every i.

We recall the following standard fact on curves:

b) If X is a smooth projective curve of genus g and Δ_X is the diagonal divisor in $X \times X$, then we have $(\Delta_X; \Delta_X) = 2 - 2g$. In particular it is negative if $g \geq 2$.

By the projection formula and property (b), we obtain:

$$\begin{aligned}(R^i_H; R^i_H) = \frac{1}{n_i} \cdot (R^i_H; R_H - \sum_{j \neq i} R^j_H) \leq & \ \tfrac{1}{n_i} \cdot (R^i_H; f^*(\Delta_X)) \\ = & \ \tfrac{1}{n_i} \cdot (f_*(R^i_H); \Delta_X) \\ = & \ \tfrac{d_i}{n_i} \cdot (\Delta_X; \Delta_X) \\ = & \ \tfrac{1}{n_i} \cdot (2 - 2g(X)).\end{aligned}$$

In particular $(R^i_h; R^i_H)$ is strictly negative.

We prove now (2): suppose that $R^i_h \neq R^j_k$. If they were numerically equivalent we would have $(R^i_H; R^i_H) = (R^i_H; R^j_k) \geq 0$. But since $(R^i_H; R^i_H) < 0$ this cannot happens.

Fix a point $q \in Y$. For every $p \in X$, the line bundles $h^*(\mathscr{O}_X(p))$ and $\mathscr{O}_Y(\deg(h) \cdot q)$ are numerically equivalent. Denote by $p_i : X \times X \to X$ and $q_i : Y \times Y \to Y$ $(i = 1, 2)$ the natural projections. We have that $f^*(p_1^a st(\mathscr{O}_X(p)) \otimes p_2^*(\mathscr{O}_X(p)))$ is numerically equivalent to $\deg(h)(q_1^*(\mathscr{O}_Y(q)) \otimes q_2^*(\mathscr{O}_Y(q)))$ and they are both ample.

Denote by $L = q_1^*(\mathscr{O}_Y(q)) \otimes q_2^*(\mathscr{O}_Y(q))$ and by $M = p_1^a st(\mathscr{O}_X(p)) \otimes p_2^*(\mathscr{O}_X(p))$.

We have that

$$\begin{aligned}(L; R^i_H) = & \ \tfrac{1}{\deg(h)} \cdot (f^*(M); R^i_H) \\ = & \ \tfrac{1}{\deg(h)} \cdot (M; f_*(R^i_H)) \\ = & \ \tfrac{d_i}{\deg(h)} \cdot (M; \Delta_X) \\ = & \ \tfrac{2 \cdot d_i}{\deg(h)}\end{aligned}$$

Since the d_i's and and $\deg(h)$ are bounded just in terms of the genus of Y, this proves (3). □

4. Lecture Four: The Mordell – Weil Theorem

In this lecture we will prove an importantTheorem on the theory of curves of genus one over a function field: The Mordell Weil Theorem:

Theorem 4.1. *(Mordell–Weil) Let E_F be a non isotrivial curve of genus one over the function field F. Suppose that $E_F(F)$ is non empty, then the set $E_F(F)$ has a natural structure of finitely generated abelian group.*

In particular observe that $E_F(F)$ is at most countable, which implies that the non isotriviality condition is necessary.

Before we give the proof, we will need to recall and explain some facts on curves and surfaces:

a) Let C be a smooth projective curve over a field k (arbitrary). Then there exists a smooth projective commutative group variety $J(C)$ called *the Jacobian of* C which classifies the divisors of degree zero on C: each k–point of $J(C)$ is a divisor of degree zero D on C *defined over* k modulo rational equivalence. The dimension of $J(C)$ is the genus of C.

b) If $O \in C$ is a k rational point, we have an algebraic map, called the *Abel Jacobi map* $j_0 : C \to J(C)$ which sends the point P in the class of $\mathscr{O}_C(P - O)$.

c) In particular, if the genus of C is one and $O \in C(k)$ is a rational point, then the Abel Jacobi map is an isomorphism. Consequently C inherit a natural structure of smooth projective group variety of dimension one. We will call the couple $(C,))$ *Elliptic curve defined over* k.

d) By definition of group variety, if the genus of C is one, as soon as we choose a k rational point $O \in C(k)$, the set $C(k)$ has a natural structure of abelian group. The Mordell Weil Theorem tells us that this, if $k = F$ and C is non isotrivial, the group is finitely generated over $\mathbf{Z}$.

e) Suppose that X_F is a a smooth projective curve and $\mathscr{X} \to B$ is a smooth projective model of it. There is a natural morphism of groups:

$$\iota : Pic(X_F) \to Pic(\mathscr{X}) \tag{4.1}$$

defined as follows: Let D_F be an irreducible divisor on X_F, the Zariski closure D of it is a curve on $\mathscr{X}$ thus a divisor on it. Then $\mathscr{O}_{X_F}(D_F) = \mathscr{O}_{\mathscr{X}}(D)$.

The map ι is well posed because the field of rational functions of X_F is the field of rational functions of $\mathscr{X}$. For the same reason, it is a group morphism.

f) In particular suppose that $(E_F, 0)$ is an elliptic curve over F. Let $f : \mathscr{E} \to B$ be a model of E_F over B, $b \in B$ and $\mathscr{E}_b$ the fibre of f over b. Suppose that cE_b is smooth, thus $\mathscr{E}_b, O_p)$ is an elliptic curve over $\mathbf{C}$. Point (f) above gives rise to a morphism of groups

$$\iota_p : E_F(F) \longrightarrow \mathscr{E}_p(\mathbf{C}). \tag{4.2}$$

It is called *morphism of specialisation at* p.

g) Let k be a field of characteristic zero and $(E, 0)$ be an elliptic curve over it. The curve E can be realised as a smooth cubic curve in P^2: Riemann Roch Theorem on E tells us that the llne bundle $\mathscr{O}_E(30)$ is very ample and $h^0(E, \mathscr{O}_E(3O)) = 3$. Thus the linear system $H^0(E, \mathscr{O}_E(3O))$ embeds E as a cubic curve in $\mathbf{P}^2$.

h) The group law on E may be described explicitly in terms of the cubic curve defined in (d) above: given two points p and q in E, we define the point $p + q$ in the following way: consider the line through p and q, it intersects the curve E in a point r; Consider the line through r and 0. the third point of intersection of it with E is $p+q$ (the only non completely evident fact that this construction equips E with the structure of a commutative group with neutral element o is the associativity).

i) The elliptic curve E has four distinct points p such that $2 \cdot p = 0$. These are O and the three points of the cubic having the tangent at the cubic which passes through 0.These points are called *points of two torsion of E*.

j) Suppose that all the points of two torsions of E are rational over k. Observe that the line passing through two of the non zero points of two torsions of E passes through the third one. Indeed the sum of two points of two torsions is again a point of two torsion thus the tangent to E in it passes through 0.

k) Suppose that all the two torsions points of E are rational over k. Impose that the line through the three non zero two torsion points is the line $Y_2 = 0$, that two of these three points have coordinates $[0:0:1]$ and $[1:0:1]$, that the point 0 is the point of homogeneous coordinates $[1:0:0]$ and that the line $Z_3 = 0$ is tangent to E in 0 with ordre of tangency equal to three (thus 0 is an inflection point of E) then equation of the elliptic curve becomes

$$Y^2 \cdot Z = X(X-Z)(X-\lambda \cdot Z) \tag{4.3}$$

In this case $\lambda \neq 0$ or 1 and the third non zero two torsion point has coordinates $[\lambda : 0 : 1]$.

l) In Lecture two we shown that every curve of genus zero over F is isotrivial, in particular it admits a model which is smooth over B. For elliptic curves too, the existence of a smooth model impies isotriviality:

Theorem 4.2. *Let (E_F, O) be an ellipitc curve over F. If E admits a projective model $f : \mathscr{E} \to B$ which is smooth over B, then E_F is isotrivial.*

In other words we can state:

Corollary 4.3. *Let E_F be a non isotrivial curve of genus one over F. Then every model $f : \mathscr{E} \to B$ of E_F is* not *smooth over B.*

Proof. (Of Theorem 4.2) We can make a finite extension of F and suppose that the two torsion points of E ar rational over F. Since all the two torsion points are rational over F, the curve E_F ca be written as in 4.3. In particular the element $\lambda \in F$ defines a function $\lambda : B \to \mathbf{P}^1$.

Fix a model $f : \mathscr{E} \to B$ which is smooth over B (we, by hypothesis, suppose that it exists).

Fix a point $b \in B$. The restriction of the specialisation morphism $\iota_p : E_F(F) \to \mathscr{E}_p(\mathbf{C})$ is surjective on the group of the two torsion points (for instance because, locally $\mathscr{E}$ is diffeomorphic to a product). Consequently, the equation of the curve $\mathscr{E}_b$ is given by $Y^2 \cdot Z = X(X-Z)(X-\lambda(b) \cdot Z)$. This implies that $\infty \notin \lambda(B)$. Consequently λ is the constant function and the conclusion follows. □

m) The Theorem above gives a way to find non isotrivial elliptic curves: For instance the curve

$$Y^2 \cdot Z = X(X-Z)(X-t \cdot Z) \tag{4.4}$$

is not isotrivial over $\mathbf{C}(t)$ (in this case the function λ is not constant).

n) Suppose that X is a smooth projective surface and $D = \sum D_i$ is a divisor over it. We will say that D is *a simple normal crossing (SNC) divisor* if D is reduced, each irreducible component of D is a smooth curve and two components of D intersect transversally.

o) Suppose that X_F is a smooth projective curve over the function field F. Let $f\mathscr{X} \to B$ be a model of X_F. There exists a open set $U_{\mathscr{X}} \subset B$ such that , for every $b \in U_{\mathscr{X}}$, the fibre $\mathscr{X}_b$ is a smooth curve. In general, if $b \notin U_{\mathscr{X}}$, the fibre $\mathscr{X}_b$ is a divisor on $\mathscr{X}$ which may *not* be SNC (it may contain non reduced or singular - or both - components).

p) Nevertheless the following Theorem holds:

Theorem 4.4. *(Semistable Reduction Theorem) Let X_F be a smooth projective curve. Then there exists a finite extension F'/F -with associated smooth projective curve B' - such that the F'–curve $X_{F'} := X_F \times_F F'$ has a model (called the Semistable model) $\mathscr{X} \to B'$ whose fibres are either smooth or SNC divisors with the following property: each irreducible component of the fibre which is a rational curve intersects the other components in at least two points.*

A proof of this theorem may be found for instance in [9] Proposition 3-48 page 118.

Proof. (of Mordell Weil Theorem) Since the theorem is true if we prove it after a finite extension of F, from Theorem 4.4, it suffices to prove the following:

Lemma 4.5. *Suppose that E_F admits a semi stable model $\mathscr{E} \to B$, then the map*

$$E_F(F) \longrightarrow Pic(E_F) \longrightarrow Pic(\mathscr{E}) \longrightarrow H^2(\mathscr{E}, \mathbf{Z}) \tag{4.5}$$

is injective.

Lemma 4.5 will be consequence of the following:

Lemma 4.6. *Let E_F as above. Let $p \in E_F(F)$ and $P : B \to \mathscr{E}$ the associated section. Then*

$$(P;P) < 0. \tag{4.6}$$

Let's see how Lemma 4.6 implies Lemma 4.5. suppose that p and q are two distinct F–rational points of E_F and P and Q be the associated sections. Then, since $(P;Q) \geq 0$ and $(P:P) < 0$, we have that $P - 0$ and $Q - O$ cannot be numerically equivalent, and consequenty their image in $H^2(\mathscr{E}, \mathbf{Z})$ are distinct. □

In order to prove Lemma 4.6 we need to know recall some more properties of the geometry of surfaces:

q) Let $f : \mathscr{X} \to B$ be a smooth projective surface fibered over the curve B. The line bundle $K_{\mathscr{X}/B} := K_{\mathscr{X}} \otimes f^*(K_B)$ (where K_X is the canonical line bundle on the variety X) is called the *relative canonical line bundle of $\mathscr{X}$ over B* and it is a model of the canonical line bundle K_{X_F} of the generic fibre X_F.

r) If $f : \mathscr{X} \to B$ is as in (f), we can define also the *relative sheaf of differentials* $\Omega_{\mathscr{X}/B}$ and we have an exact sequence (called "the first exact sequence of differentials"):

$$0 \longrightarrow f^*(\Omega_B) \longrightarrow \Omega_{\mathscr{X}} \longrightarrow \Omega_{\mathscr{X}/B} \longrightarrow 0 \tag{4.7}$$

s) As one can see from the exact sequence above, the line bundle $K_{\mathscr{X}/B}$ and the sheaf $\Omega_{\mathscr{X}/B}$ coincide over the open set of $\mathscr{X}$ where the morphism $f : \mathscr{X} \to B$ is smooth. In general there is a natural map

$$\Omega_{\mathscr{X}/B} \longrightarrow K_{\mathscr{X}/B}. \tag{4.8}$$

Indeed it is given by the natural map $\Omega_{\mathscr{X}/B} \otimes f^*(\Omega_B) \to \Lambda^2(\Omega_{\mathscr{X}}) = K_{\mathscr{X}}$ given by $[\alpha] \otimes b \to \alpha \wedge b$.

t) Suppose that $f : \mathscr{X} \to B$ is a semi stable model of the its generic fibre X_F. In this case, we can choose local equations (x, y) of $\mathscr{X}$ and t of B in a neighbourhood U of a singular point p of a fibre of f, in such a way that $f|_U : U \to f(U)$ is given by the morphism $(x, y) \to t = xy$. Thus, the restriction to U of the first exact sequence of differentials gives that $\Omega_{\mathscr{X}/B}$ is without torsion and thus the natural map 4.8 is an inclusion. Consequently, denoting by S the set of singular points in the fibres of f we get the exact sequence

$$0 \longrightarrow \Omega_{\mathscr{X}/B} \longrightarrow K_{\mathscr{X}/B} \longrightarrow \bigoplus_{p \in S} \mathbf{C}_p \longrightarrow 0. \tag{4.9}$$

Where $\mathbf{C}_p$ is the skyscraper sheaf supported on p.

u) From point (s) we get that if $\mathscr{X} \to B$ is a semi stable model of its generic fibre, we can compute the Chern classes of $\Omega_{\mathscr{X}/B}$ and obtain:

– $c_1(\Omega_{\mathscr{X}/B}) = K_{\mathscr{X}/B}$;

– $c_2(\Omega_{\mathscr{X}/B}) = s$; where s is the cardinality of S.

In order to prove that, it suffices to remark that $c_1(\bigoplus_{p \in S} \mathbf{C}_p) = 0$ and $c_2(\bigoplus_{p \in S} \mathbf{C}_p) = -s$.

v) (Intersection bilinear form on fibres) Suppose that $f : \mathscr{X} \to B$ is a smooth projective model of its generic fibre. Let $b \in B$ and $\mathscr{X}_b$ the fibre of f over it. Write the Cartier divisor $\mathscr{X}_b$ as $\mathscr{X}_b = \sum_j m_j F_j$ where F_j are irreducible. Then the intersection product on the vector space $V := \langle F_j \rangle$ is a semi defined negative form. An element $v \in V$ is in $V^\perp$ if and only if v is a multiple of the line generated by the entire fibre $\mathscr{X}_F$. In particular, for every F_j we have $(F_j; F_j) < 0$. A proof of this, which is a consequence of the Hodge Index Theorem, on surfaces, can be found for instance on [2] Corollary 2.6 page 19.

We will describe now a singular fibre of a semi stable model of a curve of genus one:

Theorem 4.7. *Suppose that $f : \mathscr{E} \to B$ is a semi stable model of its generic fibre E_F which is of genus one. Let $\mathscr{E}_b = \sum_j F_j$ be a fibre of f (observe that, since f is semi stable, the multiplicity of each F_j is one). Then each F_j is a smooth rational curve and $(F_j; F_j) = -2$ and $(K_{\mathscr{E}}; F_j) = 0$.*

Proof. We begin by remarking that

$$(F_j; F_j) = -\sum_{i \neq j} (F_j; F_i) \tag{4.10}$$

Indeed, by (n) above, $0 = (F_j; \mathscr{E}_b) = (F_j; \sum_i F_i) = (F_j; F_j) + \sum_{i \neq j} (F_j; F_i)$. Moreover , since the fibre is connected, we must have that there exists j such that $(F_i, F_j) > 0$ which implies that $(F_i, F_I) < 0$.

Observe that, the semi stability condition implies that, if F_i is a rational curve, then $(F_i, F_i) \leq -2$ (otherwise F_i would intersect the other components only in one point).

By the adjunction formula we have that $(K_{\mathscr{E}}; \mathscr{E}_b) = (K_{\mathscr{E}}; \mathscr{E}_b) + (\mathscr{E}_b; \mathscr{E}_b) = 0$. Suppose that there is F_i such that $(K_{\mathscr{E}}; F_i) \neq 0$ thus there would be another F_j such that $(K_{\mathscr{E}}, F_j) < 0$. Again by adjunction formula we have then that $(K_{\mathscr{E}}; F_j) + (F_j; F_j) \leq -2$. But this number is $2g(F_j) - 2$ which is at least -2. Thus $(K_{\mathscr{E}}, F_j) = -1$ and $(F_j; F_j) = -1$ which means that F_j is a rational curve which intersects the other components of the fibre in only one point. This is not possible for the semi stability condition. Thus we have that $(K_{\mathscr{E}}, F_j) = 0$ for any F_j.

But again, since, by adjunction, $(K_{\mathscr{E}}; F_j) + (F_j; F_j)$ is a even number which is at least -2, and $(F_j; F_j) < 0$ we must have $(F_j; F_j) = -2$ and F_j is a rational curve. □

A key step of the proof of the proof of Lemma 4.5 is the following:

Theorem 4.8. *Suppose that the curve E_F admits a semi stable model $f : \mathscr{E} \to B$. Then $f_*(K_{\mathscr{E}/B})$ is a line bundle, which we will denote by $\mathbb{L}_{\mathscr{E}}$. Moreover, the natural map*

$$f^*(\mathbb{L}_{\mathscr{E}}) \longrightarrow K_{\mathscr{E}/B} \tag{4.11}$$

is an isomorphism.

Proof. Let $U \subset B$ the subset where the morphism f is smooth. Then, the restriction of $K_{\mathscr{E}}$ to $f^{-1}(U)$ is trivial on each fibre, consequently the restriction to it of the natural map $f^*(\mathbb{L}_{\mathscr{E}})|_U \longrightarrow K_{\mathscr{E}/B}|_U$ is an isomorphism.

This implies that $K_{\mathscr{E}/B} = f^*(\mathbb{L})(\sum_j a_j F_j)$ where a_j are positive integers and F_j are irreducible components of the fibres of f.

By Theorem 4.7, we have that $(K_{\mathscr{E}/B}; F_j) = 0$ for every irreducible component $F_j$of a fibre of f. This implies, by property (u) above, that the divisor $\sum_j a_j F_j$ must be a multiple of the entire fibre, which means that all the a_i' are the same. Thus $\sum_j a_j F_j = f^*(M)$ for a suitable line bundle M on B.

But $f_*(K_{\mathscr{E}/B}) = f_*(f^*(\mathbb{L}_{\mathscr{E}}(\sum_j a_j F_j)) = \mathbb{L}_{\mathscr{E}} \otimes M$ (because $f_*(\mathscr{O}_{\mathscr{E}}) = \mathscr{O}_B$, since f is with connected fibres and projective). The conclusion follows from the definition of $\mathbb{L}_{\mathscr{E}}$. □

In order to conclude the proof of the Mordell –Weil Theorem we prove the following theorem:

Theorem 4.9. *Let $(E_F; 0)$ be an elliptic curve over F and suppose that it admits a semi stable model $f : \mathscr{E} \to B$. Then, with the notations above,*

$$\deg(\mathbb{L}_{\mathscr{E}}) = \frac{s}{12} \tag{4.12}$$

Where, as in (m) above, s is the number of singular points of the fibres of f.

As a corollary we find

Corollary 4.10. *Under the hypothesis above, if E_F is non isotrivlal, we have* $\deg(\mathbb{L}_{\mathscr{E}})) > 0$.

The Corollary is a direct consequence of the Theorem and Corollary 4.3.

Let's show how Lemma 4.6 (and thus the Mordell Weil Theorem) is a consequence of Theorem 4.9: Let $P : B \to \mathscr{E}$ be a section. Then, by adjunction formula, we have

$$0 = (P, K_{\mathscr{E},B}) + (P;P) = (P; f^*(\mathbb{L}_{\mathscr{E}})) + (P;P) = \deg(\mathbb{L}_{\mathscr{E}}) + (P;P). \tag{4.13}$$

The conclusion follows from Theorem 4.9 and the non isotriviality hypothesis.

in order to give the proof of Theorem 4.9 we need to recall some other properties of surfaces:

w) If X is a smooth projective surface, then the Noether formula holds:

$$\chi(\mathscr{O}_X) = \frac{(K_X; K_X) + c_2(X)}{12}. \tag{4.14}$$

x) (Relative Serre Duality) Suppose that $f : X \to B$ is a morphism from a smooth projective surface to a smooth projective curve. As before we define the relative canonical line bundle as $K_f := K_X \otimes f^*(K_B^{-1})$. We have then, for every vector bundle E on X, a perfect pairing of sheaves

$$f_*(E) \otimes R^1 f_*(E^\vee \otimes K_f) \longrightarrow R^1 f_*(K_f) \simeq \mathscr{O}_B. \tag{4.15}$$

y) (Leray Spectral Sequence) Suppose that $f : X \to B$ is a projective morphism from a smooth surface to a smooth curve. Let E be a vector bundle over X, Then there is a spectral sequence

$$E_2^{p,q} := H^p(B; R^q f_*(E)) \Rightarrow H^{p+q}(X, E). \tag{4.16}$$

In particular there is an exact sequence

$$0 \longrightarrow H^1(B; f_*(E)) \longrightarrow H^1(X;E) \longrightarrow H^0(B, R^1 f_*(E)) \longrightarrow 0. \tag{4.17}$$

We can now begin the proof of Theorem 4.9

Proof. (Of Theorem 4.9) From the first exact sequence of differentials (r), the Chern classes computations (t), the Noether formula (v) and the fact that $K_{\mathscr{E}} = f^*(\mathbb{L}_{\mathscr{E}} \otimes K_B)$ we obtain that

$$\chi(\mathscr{E}) = \frac{s}{12}. \tag{4.18}$$

We have that $\chi(\mathscr{E}) = h^0(\mathscr{E}, \mathscr{O}_{\mathscr{E}}) - h^1(\mathscr{E}, \mathscr{O}_{\mathscr{E}}) + h^2(\mathscr{E}, \mathscr{O}_{\mathscr{E}})$.

Let's compute each term of $\chi(\mathscr{E})$:

– $h^0(\mathscr{E}, \mathscr{O}_{\mathscr{E}}) = 1$.

– From the Leray Spectral sequence (x) we get $h^1(\mathscr{E}, \mathscr{O}_{\mathscr{E}}) = h^1(B, \mathscr{O}_B) + h^0(B, R^1 f_*(\mathscr{O}_{\mathscr{E}}))$.

– From the relative Serre duality (w) applied to $\mathbb{L}|\mathscr{E}$, we have that $R^1 f_*(\mathscr{O}_c E) = \mathbb{L}_{\mathscr{E}}^{-1}$. Consequently $h^0(B, R^1 f_*(\mathscr{O}_c E)) = h^0(B, \mathbb{L}_{\mathscr{E}}^{-1})$.

– From Serre Duality on $\mathscr{E}$, we have that $h^2(\mathscr{E}, \mathscr{O}_{\mathscr{E}}) = h^0(\mathscr{E}, K_{\mathscr{E}}) = h^0(B, \mathbb{L}_{\mathscr{E}} \otimes K_B)$. And, by Serre duality on B we obtain $h^2(\mathscr{E}, \mathscr{O}_{\mathscr{E}}) = h^1(B, \mathbb{L}_{\mathscr{E}}^{-1})$.

Thus we find, by using Riemann–Roch Theorem on B,

$$\chi(\mathscr{E}) = 1 - h^1(B, \mathscr{O}_B) -^0 (B, \mathbb{L}_{\mathscr{E}}^{-1}) + h^1(B, \mathbb{L}_{\mathscr{E}}^{-1}) = \chi(B) - \chi(\mathbb{L}_{\mathscr{E}}^{-1}) = \deg(\mathbb{L}_{\mathscr{E}}). \tag{4.19}$$

The conclusion follows. □

5. Lecture Five: Rational points on curves of higher genus

In this lecture we will prove that, if X_F is smooth, projective *non isotrivial* curve of genus bigger or equal then two over F, then the set $X_F(F)$ of its $F-$rational points is finite. The analogous statement for curves over number fields is the former Mordell Conjecture, proved by Faltings.

One can easily see that the isotriviality hypothesis is necessary: otherwise, if the curve is defined over $\mathbf{C}$ then all the points with coordinates in $\mathbf{C}$ will be F–rational.

We need to recall some other properties of surfaces:

a) *Parameter space of morphisms*: Let X be a smooth projective surface equipped with an ample line bundle L. Suppose that we have a fibration $f : X \to B$ (where B is, as in the previous lectures, a fixed smooth projective curve). Let A be a positive constant, and consider the following set

$$Hom_f(B;X)_{\leq A} := \{g : B \to X \ \text{ such that } \ f \circ g = Id_B \ \text{ and } \ \deg(g^*(L)) \leq A\}. \tag{5.1}$$

Then, there exists a quasi projective variety $\underline{Hom}_f(B;X)_{\leq A}$ such that:

– $Hom_f(B;X)_{\leq A} = \underline{Hom}_f(B;X)_{\leq A}(\mathbf{C})$. This means that every point in $\underline{Hom}_f(B;X)_{\leq A}$ "is a morphism from B to X".

– There exists a "universal" morphism

$$\begin{aligned} F : B \times \underline{Hom}_f(B;X)_{\leq A} &\longrightarrow X \\ (b,f) &\longrightarrow f(b) \end{aligned}$$

Example 5.1. Suppose that $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $B = \mathbf{P}^1$ with the natural projection on the first factor. We take as L the line bundle $\mathscr{O}(1,1)$. An element of $Hom_f(B;X)_{\leq A}$ is a map $f : \mathbf{P}^1 \to \mathbf{P}^1$ such that $f^*(\mathscr{O}(1)) = \mathscr{O}(d)$ with $0 < d \leq A$. Let d be the biggest integer less or equal to A and consider the Veronese embedding $\mathbf{P}^1 \to \mathbf{P}^d$. For every hyperplane H of codimension two of $\mathbf{P}^d$, the restriction to the image of $\mathbf{P}^1$ of the projection $pr_H : \mathbf{P}^d \dashrightarrow \mathbf{P}^1$ induced by H is an element of $Hom_f(B;X)_{\leq A}$ and every element of it is of this form. Consequently $\underline{Hom}_f(B;X)_{\leq A}$ is the Grassmannian of the hyperplanes of codimension two in $\mathbf{P}^d$.

For more details on the theory of parameter spaces of morphisms, cf. for instance [3] Chapter 3.

5.1. Introduction to height theory over function fields.

Suppose that X_F is a smooth projective variety defined over F and $f : \mathscr{X} \to B$ is a projective model of it over B. Fix a line bundle $\mathscr{L}$ over $\mathscr{X}$. If $p \in X_F(F)$ is a F–rational point and $P : B \to \mathscr{X}$ is its associated section, we define

$$h_{\mathscr{L}}(p) := \deg(P^*(\mathscr{L})) \tag{5.2}$$

and call it *the height of* p *with respect to* $\mathscr{L}$.

Given a projective variety X_F over F and a line bundle L_F over it. We will say that function $h_{\mathscr{L}}(\cdot)$ above *is a height function associated to* L_F. Thus when we fix a height function, we are implicitly fixing a model $\mathscr{X}$ of X_F and a model $\mathscr{L}$ of the line bundle L_F over it.

Of course, the value of a height function of a rational point depends on the choice of the model and of the line bundle nevertheless some important functorial proprietes hold.

1) *Addivity of heights*: If $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles as above, then, essentially by definition, for every point p we have $h_{\mathscr{L}_1\otimes\mathscr{L}_2}(p) = h_{\mathscr{L}_1}(p) + h_{\mathscr{L}_2}(p)$.

2) Suppose that $\mathscr{L} = \mathscr{O}(\mathscr{D})$ with $\mathscr{D}$ an effective divisor on $\mathscr{X}$. Suppose that p is a point such that, the associated section is not contained in $\mathscr{D}$. Then $h_{\mathscr{L}}(p) \geq 0$.

Indeed, the restriction of $\mathscr{D}$ to P is a non everywhere vanishing section of $P^*(\mathscr{L})$, thus, the degree of this is non negative. Observe that it is zero if the image of B via P do not meet $\mathscr{D}$.

3) Observe that, in the situation as in (2), in order to check that $P(B)$ is not contained in $\mathscr{D}$, it suffices to check that the point p is not contained in the restriction D_F of $\mathscr{D}$ to X_F.

Property (3) above is typical of many properties of heights: even if heights are defined and computed via models of the variety defined over F, many of their qualitative properties may be studied and checked over the variety X_F itself only. We would like to clarify this: Let's fix some terminology:

Definition 5.2. *Let B be a curve and $f : \mathscr{X} \to B$ be a morphism from a normal variety to B. Let $\mathscr{D}$ be a reduced irreducible divisor on $\mathscr{X}$:*
(i) $\mathscr{D}$ is said to be horizontal, *if $f|_{\mathscr{D}} : \mathscr{D} \to B$ is dominant;*
(ii) $\mathscr{D}$ is said to be vertical *if $f|_D : \mathscr{D} \to B$ is a point.*

Observe that every divisor $\mathscr{D} := n_i\mathscr{D}_i$ has a *unique* decomposition $\mathscr{D} = \mathscr{H} + \mathscr{V}$ with $\mathscr{H}$ which is a sum of horizontal divisors and V is a sum of a vertical divisors. The divisor H will be called the *horizontal part of* $\mathscr{D}$ and $\mathscr{V}$ will be called the *vertical part of* $\mathscr{D}$. It is very important to notice that the restriction to the generic fibre of $\mathscr{D}$ coincides with the restriction to the generic fibre of $\mathscr{H}$. Indeed, the image on B of the generic point of an irreducible vertical divisor is a closed point.

Proposition 5.3. *Suppose that $\mathscr{L}$ is a line bundle over $\mathscr{X}$ such that $\mathscr{L} = \mathscr{O}(\mathscr{V})$ where $\mathscr{V}$ is a vertical divisor. Then there exists a constant C (which is is independent on the points) such that, for every $p \in X_F(F)$ we have*

$$|h_{\mathscr{L}}(p)| \leq C. \tag{5.3}$$

Proof. Write $\mathscr{V} = \mathscr{V}_1 - \mathscr{V}_2$ where $\mathscr{V}_i$ are vertical and effective. We can find effective divisors S_i on B such that $f^*(S_i) \geq \mathscr{V}_i$. Consequently, for every rational point $p \in X_F(F)$, we have that, by property (2) above, $h_{f^*(\mathscr{O}(S_i))}(p) \geq h_{\mathscr{O}(\mathscr{V}_i)}(p) \geq 0$ (observe that, since $\mathscr{V}_i$ is vertical, $P(B)$ is not contained in it). Thus

$$-\deg(S_2) \leq h_{\mathscr{L}}(p) \leq \deg(S_1). \tag{5.4}$$

□

The proposition above tells us that, qualitatively speaking, the height function depends only on the generic fibre of the involved line bundle. More specifically:

Corollary 5.4. *Suppose that $f : \mathscr{X} \to B$ is as above. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two line bundles over $\mathscr{X}$ such that $(\mathscr{L}_1)_F \simeq (\mathscr{L}_2)_F$. Then there exists a constant C (depending on $\mathscr{L}_i$ and $\mathscr{X}$) such that, for every rational point $p \in X_F(F)$ we have*

$$|h_{\mathscr{L}_1}(p) - h_{\mathscr{L}_2}(p)| \leq C. \tag{5.5}$$

Proof. It suffices to remark that, under the hypothesis of the corollary, we have that $\mathscr{L}_1 \otimes \mathscr{L}_2^{-1} \simeq \mathscr{O}_{\mathscr{X}}(\mathscr{V})$ with $\mathscr{V}$ vertical divisor on $\mathscr{X}$ and then apply the proposition. □

We introduce now the notion of bounded set of a variety:

Definition 5.5. *Let X_F be a projective variety over a function field F. Let L_F be a line bundle defined over it. We will say that a subset $S \subset X_F(F)$ is of bounded height with respect to L_F if there exist a constant C and models $f : \mathscr{X} \to B$ of X_F and $\mathscr{L}$ of L_F over $\mathscr{X}$ for which $h_{\mathscr{L}}(p) \leq C$ for every $p \in S$.*

It is very important to observe that, for a given subset $S \subset X_F(F)$, being of bounded height with respect to L_F is a property which depends only on S and L_F and not on the chosen models:

Proposition 5.6. *Suppose that $S \subset X_F(F)$ is a subset of bounded height with respect to the line bundle L_F. Then, for every model $\mathscr{X}$ and $\mathscr{L}$ of X_F and L_F over $\mathscr{X}$ respectively, we can find a constant C (depending on the models) such that $h_{\mathscr{L}}(p) \leq C$ for every point in S.*

Proof. Since S is of bounded height with respect to L_F we can find models $\mathscr{X}_1$ and $\mathscr{L}_1$ of X_F and $\mathscr{L}$ for which $h_{\mathscr{L}_1}(p) \leq C_1$ for a suitable constant C_1.

We may suppose that $\mathscr{X}$ and $\mathscr{X}_1$ are dominated by a third model $\mathscr{X}_2$:

$$\begin{array}{ccc} & \mathscr{X}_2 & \\ {}^{\alpha}\swarrow & & \searrow^{\alpha_1} \\ \mathscr{X} & \dashrightarrow & \mathscr{X}_1 \end{array} \tag{5.6}$$

Each point $p \in X_F)$ extends uniquely to sections $P : B \to \mathscr{X}$, $P_1 : B \to \mathscr{X}_1$ and $P_2 : B \to \mathscr{X}_2$. The conclusion follows from the fact that $\alpha^*(\mathscr{L})$ and $\alpha_1^*(\mathscr{L}_1)$ are both models of L_F over $\mathscr{X}_2$ and $\alpha \circ P_2 = P$ and $\alpha_1 \circ P_2 = P_1$. □

A similar proof gives a functoriality property of heights:

Proposition 5.7. *Suppose that $f : X_F \to Y_F$ is a morphism between projective varieties over F. Let L_F be a line bundle over Y_F. For every height functions $h_{\mathscr{L}}(\cdot)$ and $h_{f^*(L_F)}(\cdot)$ associated to L_F and $f^*(L_F)$ respectively, we can find a constant (depending on the choice of the height functions) such that, for every rational point $p \in X_F(F)$ we have*

$$|h_{\mathscr{L}}(f(p)) - h_{f^*(L_F)}(p)| \leq C. \tag{5.7}$$

The proof, which is essentially the same of the proof of Proposition 5.6, is left as an exercise to the reader.

One can also refine Proposition 5.7 and see that the property of being of bounded height is actually independent of the chosen line bundle:

Proposition 5.8. *Let S be a subset of $X_F(F)$ and L_1 and L_2 be two ample line bundles on X_F. Then S is of bounded height with respect to L_1 if and only if it is of bounded height with respect to L_2.*

In order to prove this Proposition, we prove first a Lemma which is interesting in its own:

Lemma 5.9. *Suppose that L_F is a line bundle which is generated by global sections on the projective variety X_F. Then for every projective model $f : \mathscr{X} \to B$ of X_F and every model $\mathscr{L}$ of L_F over it we can find a constant C (depending on the models), such that, for every rational point $p \in X_F(F)$ we have*

$$h_{\mathscr{L}}(p) \geq C. \tag{5.8}$$

Proof. It suffices to prove that there exists a model $f : \mathscr{X} \to B$ of X_F and a model $\mathscr{L}$ of L_F over it for which the property of the Lemma holds.

Since L_F is globally generated, the linear system $H^0(X_F, L_F)$ give rise to a morphism $\varphi_L : X_F \to \mathbf{P}^N$ for a suitable N. The Zariski closure $\mathscr{L}$ of X_F in $B \times \mathbf{P}^N$ is projective model of X_F with a line bundle $\mathscr{L}$ (the pull back of $cO_{\mathbf{P}}(1)$) which is a model of L_F and is globally generated. This means that we can find a basis $s_0, \dots, s_r$ of $H^0(\mathscr{X}, \mathscr{L})$ with the property that for every subvariety Z of $\mathscr{X}$, we can find at least one of the s_i's which do not vanish on it. Property (2) above allows to conclude. □

The proof ot 5.8 is now straightforward:

Proof. (*of Proposition 5.8)* Since L_i are ample, we can find a positive constant n such that $L_1^{\otimes n} \otimes L_2^{-1}$ and $L_2^{\otimes n} \otimes L_1^{-1}$i are both generated by global sections. Thus, for every model $\mathscr{X}$ of X_F and models $\mathscr{L}_1$ and $\mathscr{L}_2$ of L_i over it, we can find a constants C depending on the models, such that, for every rational point $p \in X_F(F)$ we have

$$n \cdot h_{\mathscr{L}_1}(p) - h_{\mathscr{L}_2}(p) \geq C \qquad n \cdot h_{\mathscr{L}_2}(p) - h_{\mathscr{L}_1}(p) \geq C. \tag{5.9}$$

The conclusion easily follows. □

From Proposition 5.8 we can call a subset $S \subset X_F(F)$ of bounded height with respect to an ample line bundle just *"a bounded set of $X_F(F)$"*.

An important consequence of Lemma 5.9 is the following:

Corollary 5.10. *Suppose that L_F is an ample line bundle on the projective variety X_F. Then for every projective model $f : \mathscr{X} \to B$ of X_F and every model $\mathscr{L}$ of L_F over it we can find a constant C (depending on the models), such that, for every rational point $p \in X_F(F)$ we have*

$$h_{\mathscr{L}}(p) \geq C. \tag{5.10}$$

Proof. It suffices to remark that, since L_F is ample, for N big enough, the line bundle L_F^N is generated by global sections and $h_{\mathscr{L}^N}(\cdot) = N \cdot h_{\mathscr{L}}(\cdot)$. The conclusion follows from Lemma 5.9. □

We can now state and prove the Theorem which relates the height theory to the finiteness properties of rational points on curves:

Theorem 5.11. *Let X_F be a smooth projective curve over a function field F. Let L_F be a an ample line bundle over X_F. Suppose that there exists a subset $S \subset X_F(F)$ and which is bounded and infinite. Then X_F is isotrivial.*

A similar Theorem holds for X_F of genus one but since we will not need it here, we will restrict our proof to the case of higher genus curves. Observe also that if X_F is of genus zero, then, since $X_F \simeq \mathbf{P}^1$ (lecture three), one can easily find infinitely many rational points on it of bounded height.

Proof. Fix a regular projective model $f : \mathscr{X} \to B$ of X_F and an ample line bundle $\mathscr{L}$ over it. By definition, we can find an infinite set of sections of f

$$P : B \longrightarrow \mathscr{X} \tag{5.11}$$

such that $\deg(P^*(\mathscr{L})) \leq C$, for a suitable constant C independent on P.

By property (a), we can consequently find a quasi projective variety $\underline{Hom}_f(B, \mathscr{X})_{\leq C}$ and a *dominant morphism*

$$F : B \times \underline{Hom}_f(B, \mathscr{X})_{\leq C} \longrightarrow \mathscr{X} \tag{5.12}$$

Indeed, the restriction to the generic fibre of F contains the subset S which, being infinite, is Zariski dense.

Since $\underline{Hom}_f(B, \mathscr{X})_{\leq C}$ is quasi projective, We can cut it with hyperplanes and obtain a quasi projective (consequently affine or projective) curve Y (defined over $\mathbf{C}$) with a dominant morphism

$$F|_Y : B \times Y \longrightarrow \mathscr{X}. \tag{5.13}$$

Let $\overline{Y}$ be the normalization of the projective compactification of Y. The restriction to the generic fibre of the morphism F_Y extends to a morphism $\overline{F} : \overline{Y} \to X_F$. Since, by construction, $\overline{Y}$ is isotrivial, the conclusion follows from Theorem 3.4. □

5.2. **Mordell conjecture over function fields.** We can now start the proof of the Mordell conjecture over F. We begin by state it:

Theorem 5.12. *Let X_F be a smooth projective curve of genus at least two defined over the function field F. Suppose that X_F is* not *isotrivial. Then the set $X_F(S)$ is a bounded set. In particular it is finite.*

In order to prove the Theorem we can make some reductions and recall some further properties of vector bundles on surfaces:

– We first remark that the finiteness property follows from the boundedness property, the non isotriviality and Theorem 5.11.

– We can make a base extension and we can suppose that X_F has a regular semi stable model.

We recall now some properties of vector bundles over curves:

a) Let E be a vector bundle of rank r over a variety X. We can associate to it a projective bundle $p : \mathbf{P}(E) \to X$ which is a projective morphism whose fibers are $\mathbf{P}^{r-1}$.

Over $\mathbf{P}(E)$ we have a tautological line bundle $\mathscr{O}_P(1)$ with a surjection

$$p^*(E) \longrightarrow \mathscr{O}_P(1). \tag{5.14}$$

b) Let T be a variety. To give a morphism $g : T \to \mathbf{P}(E)$ is equivalent to give a couple $(h; L)$ where $h : T \to X$ is a morphism, L is a line bundle on T equipped with a surjection of sheaves $h^a st(E) \to L$. Moreover, in this case, $g^*(\mathscr{O}_{\mathbf{P}}(1)) = L$.

c) A vector bundle E on X is said to be *ample* if the line bundle $\mathscr{O}_P(1)$ is ample on $\mathbf{P}(E)$.

d) If E is a rank two vector bundle over a curve X, the surface $\mathbf{P}(E)$ is called *ruled surface with ruling* X. In this case $(\mathscr{O}_P(1); \mathscr{O}_P(1)) = \deg(\bigwedge^2(E))$.

e) An exact sequence of vector bundles (over an arbitrary variety)

$$\mathscr{E} :\ 0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ j\ } B \longrightarrow 0 \tag{5.15}$$

is said to be *split*, if we can find a morphism $\alpha : B \to E$ such that $j \circ \alpha : B \to B$ is the identity. This is equivalent to say that the morphism

$$A \oplus B \xrightarrow{(i,j)} E \tag{5.16}$$

is an isomorphism.

f) If we tensorise the exact sequence 5.18 by $B^\vee$ (the dual of B) and we take the associated cohomology sequence we obtain

$$H^0(X; End(B)) \longrightarrow H^1(X, A \otimes B^\vee) \tag{5.17}$$

$$Id_B \longrightarrow \delta_{\mathscr{E}}.$$

The class $\delta_{\mathscr{E}} \in H^1(X, A \otimes B^\vee)$ vanishes if and only if the exact sequence splits.

Proposition 5.13. *Suppose we have an exact sequence as in 5.18 over a smooth projective curve X. Let $f : Y \to X$ be a possibly ramified covering of curves. The exact sequence 5.18 rises to an exact sequence*

$$f^*(\mathscr{E}) :\ 0 \longrightarrow f^*(A) \xrightarrow{\ i\ } f^*(E) \xrightarrow{\ j\ } f^*(B) \longrightarrow 0 \tag{5.18}$$

Then $f^*(\mathscr{E})$ splits if and only if $\mathscr{E}$ splits.

Proof. If $\mathscr{E}$ splits, then it is evident that $f^*(\mathscr{E})$ splits too.

Suppose that $f^*(\mathscr{E})$ splits. This means that the class $\delta_{f^*(\mathscr{E})} \in H^1(Y; f^*(A \otimes B^\vee))$ vanishes.

The natural inclusion $\mathscr{O}_X \to f_*(cO_Y)$ has a natural spllitting given by the trace. Thus for every vector bundle G on X the natural map $f^* : H^1(X; G) \to H^1(X; f_*(f^*(G))) = H^1(Y; f^*(G))$ is an inclusion (observe that the last equality is due to the fact that the morphism f is finite).

A simple diagram chasing shows that the image, via f^*, of $\delta_{\mathscr{E}}$ in $H^1(Y; f^*(A \otimes B^\vee))$ is $\delta_{f^*(cE)}$. Since this last class vanishes by hypothesis, the injectivity of f^* implies the conclusion. □

Theorem 5.14. *Let X be a smooth projective curve and E be a vector bundle of rank two over it. Suppose that we have an fits an exact sequence*

$$\mathscr{E}\ :\ 0 \longrightarrow \mathscr{O}_Y \xrightarrow{\alpha} E \xrightarrow{\gamma} L \longrightarrow 0 \tag{5.19}$$

with L a line bundle of positive degree. Then E is ample if and only if the exact sequence is non split.

Proof. We first remark that $\bigwedge^2(E) = L$ thus $(\mathscr{O}_P(1); \mathscr{O}_P(!)) = \deg(L) > 0$ by hypothesis.

If $\mathscr{E}$ is split, then $\mathscr{O}_P(1)$ cannot be ample. Indeed, $E \simeq \mathscr{O}_X \oplus L$; consequently, the surjection $E \to \mathscr{O}_X$ given by the first projection give rise to an embedding $s : X \to \mathbf{P}(E)$ such that $s^*(\mathscr{O}_P(1)) = \mathscr{O}_X$. In particular $(\mathscr{O}_P(1); s(X)) = 0$. So by Nakai- Moishezon criterion, $\mathscr{O}_P(1)$ is not ample in this case.

Suppose now that $\mathscr{E}$ is non split. Let Y be a curve contained in $\mathbf{P}(E)$. Let $\iota : Y \to X$ the projection. By functoriality, we have a surjection $\iota^*(E) \to \mathscr{O}_O(1)|_Y$. If ι is not surjective, then Y is a fibre of the projective bundle and $\deg(\iota(\mathscr{O}_P(1)|_Y)) = 1 > 0$. Otherwise $\iota : Y \to X$ is a finite covering and we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_Y & \xrightarrow{\alpha} & \iota^*(E) & \longrightarrow & \iota^*(L) & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle\beta} & & & & \\ & & & & M & & & & \end{array} \tag{5.20}$$

Where M is a line bundle over Y. Moreover $(\mathscr{O}_P(1); Y) = \deg(M)$.

The degree of M cannot be negative: indeed, either the morphism $\beta \circ \alpha : \mathscr{O}_Y \to M$ is non zero, and consequently M is effective, or $\beta \circ \alpha$ is the zero morphism, and, in this case, we have an injection of $\iota^*(L)$ in M, thus $\deg(M) \geq \deg(\iota^*(L)) > 0$.

Suppose that $\deg(M) = 0$. In this case, the morphism of line bundles $\beta \circ \alpha$ cannot be zero. Thus it is an isomorphism of line bundles, which implies that the morphism

$$(\beta; \gamma) : E \longrightarrow \mathscr{O}_Y \oplus \iota^*(L) \tag{5.21}$$

is a splitting of $\iota^*(\mathscr{E})$. But, by Proposition 5.13, this is not possible. □

We fix a semi regular projective stable model $f : \mathscr{X} \to B$ of our curve X_F.

The first exact sequence of differentials give rise to an exact sequence

$$\mathscr{E}_\Omega\ :\ 0 \longrightarrow f^*(\Omega_B) \longrightarrow \Omega_{\mathscr{X}} \longrightarrow \Omega_{\mathscr{X}/B} \longrightarrow 0 \tag{5.22}$$

(cf. (r) of the previous Lecture). We can restrict the exact sequence to the generic fibre and we get an exact sequence over X_F:

$$(\mathscr{E}_\Omega)_F\ :\ 0 \longrightarrow \mathscr{O}_{X_F} \longrightarrow \mathscr{E} \longrightarrow \Omega_{X_F} \longrightarrow 0. \tag{5.23}$$

The exact sequence above is a fundamental tool in the study of diophantine properties of varieties (curves in these lectures) over function fields. We would like to remark that. a similar exact sequence is not available (at the moment) for varieties defined over a number field. This is one of the reasons why the diophantine geometry over number fields is much more involved than diophantine geometry over function fields.

Each rational point $p \in X_F(F)$ gives rise to a section $P : B \to \mathscr{X}$. The morphism of differentials give rise to a surjection

$$dP : P^*(\Omega_{\mathscr{X}}) \longrightarrow \Omega_B \tag{5.24}$$

thus to a "derivation map"

$$\begin{array}{ccc} & & \mathbf{P}(\Omega_{\mathscr{X}}) \\ & \overset{P'}{\nearrow} & \downarrow \\ B & \xrightarrow{\quad P \quad} & \mathscr{X} \end{array} \tag{5.25}$$

And moreover $(P')^*(\mathscr{O}_P(1)) = \Omega_B$ and consequently

$$\deg((P')^*(\mathscr{O}_P(1))) = 2g(B) - 2. \tag{5.26}$$

In order to conclude the proof of Theorem 5.12 and consequently of the Mordell conjecture over function fields, we should analyse the case when the exact sequence $(\mathscr{E}_\Omega)_F$ is split or not.

5.3. **The exact sequence $(\mathscr{E}_\Omega)_F$ is non split.** In this case, by Theorem 5.14, the line bundle $\mathscr{O}_P(1)$ is ample on $p : \mathbf{P}(\mathscr{E}) \to X_F$.

Let L_F be an ample line bundle on X_F. For N sufficiently big, the line bundle $\mathscr{O}_P(N) \otimes p^*(L_F^{-1})$ is ample on $\mathbf{P}(\mathscr{E})$.

The projective bundle $p_{\mathscr{X}} : \mathbf{P}(\Omega_{\mathscr{X}}) \to \mathscr{X}$ is a model of $p : \mathbf{P}(\mathscr{E}) \to X_F$. We may suppose that L_F extends to an ample line bundle over $\mathscr{X}$.

By Corollary 5.10 we can find a constant C such that, for every rational point $q \in \mathbf{P}(\mathscr{E})$ we have

$$h_{\mathscr{O}(N) \otimes p^*(\mathscr{L}^{-1})}(q) \geq C \tag{5.27}$$

Consequently from the functioriality of heights, Proposition 5.7 and formula 5.26, we obtain that, for every rational point $p \in X_F(F)$,

$$h_{\mathscr{L}}(p) \leq N \cdot (2g(B) - 2) + C. \tag{5.28}$$

This, by Theorem 5.7 ends the proof of Theorem 5.12 in this case, because F_F is supposed to be non isotrivial.

5.4. **The exact sequence $(\mathscr{E}_\Omega)_F$ is split.** In this case we will prove the following:

Theorem 5.15. *The exact sequence $(\mathscr{E}_\Omega)_F$ is split if and only if the curve X_F is isotrivial.*

In order to prove Theorem 5.15 above, we need to recall some facts about foliations on surfaces.

g) Let $\mathscr{X}$ be a smooth surface (not necessarily projective). A *regular foliation $\mathscr{F}$ on $\mathscr{X}$* is a sub line bundle $N_{\mathscr{F}}$ of the cotangent bundle $\Omega_{\mathscr{X}}$. Observe that the quotient $K_{\mathscr{F}} := \Omega_{\mathscr{X}}?N$ is also locally free of rank one. Thus we have an exact sequence

$$0 \longrightarrow N_{\mathscr{F}} \longrightarrow \Omega_{\mathscr{X}} \longrightarrow K_{\mathscr{F}} \longrightarrow 0 \tag{5.29}$$

h) let M be a Riemann surface (not necessarily compact nor algebraic). A morphism $\iota : M \to \mathscr{X}$ is said to be *a leaf of the foliation $\mathscr{F}$* if:

h.1) The morphism ι is an embedding;

h.2) the natural map $\iota^*(N_{\mathscr{F}}) \to \iota^*(\Omega_{\mathscr{X}}) \to \Omega_M$, is the zero map.

In particular $\Omega_M = \iota^*(K_{\mathscr{F}})$.

i) if $z \in \mathscr{X}$ is a point, then the is an unique leaf of the foliation passing through it.

j) Denote by Δ the one dimensional unit disk. If $z \in \mathscr{X}$, then there is an *analytic* neighborhood $z \in U \subset \mathscr{X}$ isomorphic to $\Delta \times \Delta$ with coordinates (z_1, z_2) and the restriction of the exact sequence 5.29 to U is the exact sequence

$$0 \longrightarrow \mathscr{O}_U dz_1 \longrightarrow \mathscr{O}_U dz_1 \oplus \mathscr{O}_U dz_2 \longrightarrow \mathscr{O}_U dz_2 \longrightarrow 0. \tag{5.30}$$

Consequently the leaves of the foliation passing through U are given by the equations $z_1 = c$ $(c \in \Delta)$.

Actually, near a point $z \in \mathscr{X}$, a foliation is just a differential form $\omega = f_1(z_1, z_2)dz_1 + f_2(z_1, z_2)dz_2$ with $f_1(z_1, z_2)$ or $f_2(z_1, z_2$ (or both) non vanishing at z. Thus the foliation defines a local differential equation on $\mathscr{X}$. The local solutions of the differential equations give the decomposition explained in (j).

We can now give the proof of Theorem 5.15:

Proof. (Of Theorem 5.15) Let U be the open set of B where the morphism $f|_U : \mathscr{X}|_U \to U$ is smooth. The exact sequence $\mathscr{E}_\Omega)_F$ is the restriction to the generic fibre ot the exact sequence

$$(\mathscr{E}_\Omega)_U \ : \ 0 \longrightarrow f^*(\Omega_U) \longrightarrow \Omega_U \longrightarrow \Omega_{\mathscr{X}|_U/U} \longrightarrow 0. \tag{5.31}$$

Since $f|_U$ is a smooth morphism of relative dimension one, the sheaf $\Omega_{\mathscr{X}|_U/U}$ is a line bundle on $\mathscr{X}|_U$.

The exact sequence 5.31 give rise to a class $\delta_{(\mathscr{E}_\Omega)_U} \in H^1(\mathscr{X}|_U; (\Omega_{\mathscr{X}|_U/U}))^\vee)$ (property (f)). Moreover the natural inclusion $\iota : X_F \to \mathscr{X}|_U$ gives rise to map $\iota^* : H^1(\mathscr{X}|_U; (\Omega^\vee_{\mathscr{X}|_U/U})) \to H^1(X_F; (\Omega_{X_F})^\vee)$ (observe that $\iota^*(\Omega_{\mathscr{X}|_U/U}) = \Omega X_F$). This map is injective. Indeed, $f_*(\Omega^\vee_{\mathscr{X}|_U/U}) = 0$, thus by the Leray spectral sequence (y) of previous lecture, we have an inclusion

$$H^1(\mathscr{X}|_U; (\Omega^\vee_{\mathscr{X}|_U/U})) \hookrightarrow H^0(U, R^1 f_*(\Omega^\vee_{\mathscr{X}|_U/U}))$$

and $R^1 f_*(\Omega^\vee_{\mathscr{X}|_U/U})$ is without torsion, thus locally free, consequently it injects in its local fibre.

By diagram chasing we see that $\iota^*(\delta_{(\mathscr{E}_\Omega)_U}) = \delta_{(\mathscr{E}_\Omega)_F}$. Thus, by property (f), the exact sequence $(\mathscr{E}_\Omega)_F$ splits if and only if the exact sequence $(\mathscr{E}_\Omega)_U$ splits.

Consequently, from the hypothesis, the exact sequence $(\mathscr{E}_\Omega)_U$ splits.

So we have a foliation

$$\mathscr{F} \ : \ 0 \longrightarrow \Omega_{\mathscr{X}|_U/U} \longrightarrow \Omega_U \longrightarrow f^*(\Omega_U) \longrightarrow 0. \tag{5.32}$$

Fix a point $b \in U$ and the fibre $\mathscr{X}_b$ over it. For every point $z \in \mathscr{X}_b$, let $\mathscr{V}_z \simeq \Delta \times \Delta$ the neighbourhood of z in $\mathscr{X}_U$ which trivializes the foliation $\mathscr{F}$ as in (j) above. If the coordinates of z in $\mathscr{V}$ are $(0,0)$, a local inspection shows that the equation of $\mathscr{X}_b \cap \mathscr{V}_z$ is given by $z_2 = 0$ (and the leaves are given by $z_1 = c$). Moreover the function $f_{\mathscr{V}_z} : \mathscr{V}_z \to U$ is given by $(z_1, z_2) \to z_2$. In particular, for every $c \in \Delta$, the restriction of f to $\{c\} \times \Delta$ is an isomorphism with its image.

Since $\mathscr{X}_b$ is compact, we can cover it with finitely many open sets of the form $\mathscr{V}_z$ with $z \in \mathscr{X}_b$. Call $\mathscr{W}_b$ the open set union of these $\mathscr{W}_z$'s.

We can choose an ϵ and a disk Δ_ϵ of radius ϵ centred in b such that $\mathscr{V}_b := f^{-1}(\Delta_\epsilon)$ is contained in $\mathscr{W}_b$. The open set $\mathscr{V}_b$ is a "tubular neighbourhood of the curve $\mathscr{X}_b$.

Let $b_1 \in \Delta_\epsilon$ and denote by $\mathscr{X}_{b_1}$ the fibre of f over b_1. By construction, for every $z \in \mathscr{X}_b$ the leaf of $\mathscr{F}$ passing through z meets $\mathscr{X}_{b_1}$ in a unique point $\alpha(z)$ and moreover the map $\alpha : \mathscr{X}_b \to \mathscr{X}_{b_1}$ sending z in $\alpha(z)$ is analytic and surjective.

Consequently, for every $b_1 \in \Delta_\epsilon$ the curve $\mathscr{X}_{b_1}$ is isomorphic to $\mathscr{X}_b$.

We can now conclude by using property (a) of Lecture 3:

Consider the scheme $Isom_B(\mathscr{X}; B \times \mathscr{X}_b) \to B$. It is finite over B and dominant. Indeed, the image contains the open set Δ_ϵ which is dense for the Zariski topology.

Thus, there exists an extension F_1 of the field F such that $Isom_F(X_F; \mathscr{X}_1)(F_1)$ is non empty, which exactly means that X_F is isotrivial, $\square$

6. Conclusions

6.1. **Back to arithmetic.** The main results of these notes hold also for curves defined over number fields. Nevertheless for curves of genus zero one have to be careful:

– As remarked before, one can find conics (thus curves of genus zero) defined over $\mathbf{Q}$ which do not have rational points. Thus these curves are curves of genus zero which are *not* isomorphic (over $\mathbf{Q}$) to the projective line. The following Theorem is the analogue of Theorem 2.4 over a number field:

Theorem 6.1. *Let X be a conic defined over a number field K. Suppose that for every place $\mathfrak{p}$ of K (finite or infinite), we have that $X(K_\mathfrak{p}) \neq \emptyset$ then $X(K) \neq \emptyset$.*

As explained before, this Theorem is optimal. Strictly speaking, if one would like a similar analogue over function fields, it suffices to remark that, each place of a function field F corresponds to a closed point of the curve B and the restriction of a conic to it is not empty (because of Hensel Lemma and the fact that every curve has a point over $\mathbf{C}$). Thus the hypothesis of the analogue of Theorem 6.1 are always verified over a function field and consequently we find that every conic has a point.

– The analogue of Theorem 4.1 holds over a number field:

Theorem 6.2. *Let $(E; O)$ be a smooth projective curve of genus one equipped with a rational point over a number field K. Then the set E has a natural structure of commutative group variety (with trivial element 0) and the group $E(K)$ is finitely generated.*

This theorem is usually called *Mordell–Weil Theorem* and remark that there is no "non isotriviality" hyposesis involved. Actually, at the moment, we do not know what an analogue of isotrivial varieties should be over a number field.

– A Theorem analogue to Theorem 5.12 holds over number fields:

Theorem 6.3. *Let X be a smooth projective curve over a number field K. Then the set $X(K)$ is finite.*

The proof of this Theorem is much more involved (G. Faltings obtained the Fields Medal for the proof of it). Observe that again there is no "non isotriviality" hypothesis. At the moment, we are not able to bound "geometrically" the height of $K-$ rational points of X. One should remark that in the function field case, one can geometrically bound the height of rational points of a curve.

6.2. **Research directions.** There are two main streams of research in the theme of rational points of varieties over function fields (and similarly over number fields):

– One open problem, which is motivated by some deep conjectures in higher dimension is the so called *Uniform Bound Conjecture*:

Conjecture 6.4. *Let F be a function field in one variable (in characteristic zero). Then there exists a constant $N = N(g, F)$ depending only on F and g such that, for every non isotrivial smooth projective curve F_X of genus at least two defined over F, one has*

$$Card(X_F(F)) \leq N.$$

The best known result in this direction is obtained by Caporaso [4].

– Another research direction is the study of rational points of higher dimensional varieties. In this case one expects that the classification of the projective varieties can give informations on the structure of rational points. The naïve version of conjectures may easily lead to contradictions, cf. for instance [6], but there are deep conjectures which give some interesting insights on the possible future researches. We refer to [1] for more details.

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