SHIING-SHEN CHERN: A GREAT GEOMETER OF 20$^{\text{TH}}$ CENTURY

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At the Center of Mathematical Sciences and Applications at Harvard University, we have launched a new lecture series called "The History and Literature of Mathematics." The purpose of these lectures is to introduce key developments in various branches of modern mathematics to students and scholars so that they can gain a global view of the subject and hopefully acquire enough knowledge to be able to cooperate with scientists working in different though related fields. I am delivering the first of these lectures, which fittingly concerns Shiing-Shen Chern–my former advisor, someone who was an important mentor to me and to many other mathematicians over many generations, as well as a leading figure in the field of geometry.

1. Introduction

In 1675, Isaac Newton (1643-1721) said in an oft-repeated remark: "If I have seen further, it is by standing upon the shoulders of giants."

Chern is, himself, a giant of twentieth century geometry upon whose shoulders many later practitioners in the field have stood, I among them. Of course, Chern also stood on the shoulders of several great geometers before him. Among the mathematicians he considered most influential were Wilhelm Blaschke, Erich Kähler, Élie Cartan, and André Weil. Blaschke, Kähler, and Cartan taught him projective differential geometry, integral geometry, Kähler geometry, the Cartan–Kähler system, the theory of connections, and Schubert calculus, while his friend Weil encouraged him to find an intrinsic proof of the Gauss–Bonnet formula and to study characteristic classes [22].

I believe it is instructive to identify the key figures of the nineteenth century whose ideas inspired Chern and other great geometers in the next century. The study of differential invariants, for instance, can be traced back to Bernhard Riemann, Elwin Bruno Christoffel, Gregorio Ricci-Curbastro, Tullio Levi-Civita, and Hermann Weyl. The theory of Cartan–Kähler had a direct bearing on a large portion of Chern's work. His efforts on the Gauss–Bonnet formula, constructions of Chern forms, Chern–Bott forms, Chern–Moser invariants, and Chern–Simons invariants offer good examples. Integral geometry and the geometry of line complexes and Grassmannians also played pivotal roles in Chern's construction of Chern classes and in understanding the cohomology of the classifying space of vector bundles. I shall therefore begin by discussing some of the work carried out by the luminaries who preceded Chern.

2. The foundation of modern geometry in the nineteenth century

In the nineteenth century, developments in geometry had three major directions, apart from work in variational calculus that had an even longer history, evolving since Euler's time:

1. Intrinsic geometry developed by Riemann (1826-1866) based on the equivalence principle (although Riemann did not, so far as I'm aware, use that term to explicitly describe his contributions in this area).
2. Geometry based on the study of the relevant family of linear subspaces.
3. Symmetries in geometry.

We shall say more about these directions in the following sections.

2.1. **Intrinsic geometry developed by Riemann.** After Newton introduced calculus to facilitate the study of mechanics, it was soon employed by Leonhard Euler (1707-1783) for a different purpose, namely to study geometry. Euler confined his inquiries to surfaces in Euclidean spaces. But his view was similar to that held by Newton: The universe is static, and we, therefore, can use a global Cartesian coordinate system to measure everything. This view was changed drastically by Riemann. He, in turn, was influenced by his teacher Carl Friedrich Gauss (1777-1855) who noted, among other things, that Gauss curvature is intrinsic.

Riemann set many ambitious goals for himself:

- To establish a concept of space that is independent of the choice of coordinate systems.
- To explore the foundation of physics over such an intrinsically defined space by its metric tensor and its topology.
- To obtain global information about a space by linking geometry with topology.

Riemann was motivated by a grand ambition: He wanted to understand the physical world through geometry. His labors toward this end were guided by a kind of equivalence principle, which held that the essential laws of physics and geometry should be independent of the choice of coordinate system and independent of the observer's point of view. In other words, the essential geometric properties of a space, or of an object sitting within that space, should not change, regardless of whether we use Cartesian coordinates or polar coordinates to calculate them. This principle of equivalence

The principle of equivalence[1] is also the same guiding principle that laid the foundation for the general theory of relativity constructed by Albert Einstein 60 years later.

Riemann's fundamental achievements were released to the world in 1854 during his now-classic lecture on curved space called, "On the hypotheses that lie at the foundation of geometry." (The facts about Riemann's work reported here appeared in the *Collected Papers* of Bernhard Riemann, edited and translated by Roger Baker, Charles Christenson, and Henry Orde and published by Kendrick Press.) The great desire by Riemann to bring this version of the equivalence principle into geometry drove him to develop methods for determining the conditions under which two differential quadratic forms can be shown to be equivalent to each other by coordinate transformations.

Riemann introduced the curvature tensor in an essay that was written to provide an answer to the prize question on heat distribution posed by the Paris Academy. His essay, submitted July, 1861, was accompanied by a motto, written in Latin, which roughly translated to: "These principles pave the way to higher things." In this paper, Riemann wrote down the curvature tensor of the differential quadratic form, considering the equivalence of two differential quadratic forms to be a necessary condition. Both the concepts of a tensor and the intrinsic curvature tensor were brand new ideas that Riemann had intended to develop further but was not able to because of illness. And despite the tremendous importance of Riemann's essay—certainly as viewed from a contemporary lens-he did not receive the prize from the Paris Academy, nor did anyone else that year.

Heinrich Martin Weber (1842-1913) explained these ideas in greater detail based on an unpublished paper of Richard Dedekind (1831-1916) in 1867. From 1869 to 1870, Christoffel (1829-1900) and Rudolf Lipschitz (1832-1903) discussed the curvature tensor further (in research published in the *Crelle* mathematics journal), noting that they had provided sufficient conditions for the equivalence of two differential quadratic forms.

Levi-Civita (1873-1941) and Ricci (1853-1925) presented their theory of tensors in a 1901 paper, "Methods de calcul differential absolute et leur applications" (*Matematische Annalen* 54. 1901, 125-201), in which they introduced the Ricci tensor.

In 1902, Luigi Bianchi (1856-1928) published the conservation law for the Ricci tensor in a paper called "Sui simboli a quattro indici e sulla curvatura di Riemann."

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[1]It is a concept initiated in early 17th century by Galileo when he observed, by experiment, that the acceleration of a test mass due to gravitation is independent of the amount of mass being accelerated. Kepler, based on the equivalence between gravity and inertia, described what would occur if the moon were stopped in its orbit and dropped towards earth, without knowing in what manner gravity decreases with distance. Newton then deduced Kepler's plater laws: how gravity reduces with distance. The trajectory of a point mass in a gravitational field depends only on its initial position and velocity, and is independent of its composition and structure. Einstein said: "The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime." Einstein principle of relativity implies that the outcome of local experiments must be independent of the velocities of the apparatus. In particular dimensionless physical values such as the fine structure constant or electron to proton mass ratio cannot depend on where in space and time we measure them.

Bianchi concluded that the conservation law could be satisfied by subtracting from the Ricci tensor the following term, $R/2$ multiplied by the metric tensor. This modified tensor was utilized by Einstein and David Hilbert (1862-1943) in the field equations of general relativity that they completed independently at around the same time, both submitting papers for publication in late November 1915. But Einstein, alone, deserves credit for formulating an entirely new way of thinking about gravity–a decade-long effort on his part that led to a breakthrough in our understanding of the universe as a four-dimensional amalgam of space and time. That said, the critical contributions to this breakthrough made by mathematicians-including Riemann, Hilbert, Hermann Minkowski, Marcel Grossman, Levi-Civita, Ricci, and many others—should not be ignored.

In a separate direction, Riemann initiated a new approach, involving the use topology in complex analysis through the concept of Riemann surfaces. He uncovered a deep relationship between analysis and the global topology of Riemann surfaces (as spelled out in the Riemann–Roch formula), which showed how to calculate the dimension of meromorphic functions with prescribed poles in terms of topological data. Riemann started to develop the concept of "handle body decomposition," which paved the way for Henri Poincaré's work on topology and global analysis on the manifold. As Riemann stated:

"In the course of our presentation, we have taken care to separate the topological relations from the metric relations. We found that different measurement systems are conceivable for one and the same topological structure, and we have sought to find a simple system of measurements, which allows all the metric relations in this space to be fully determined and all metric theorems applying to this space to be deduced as necessary conditions."

Riemann was puzzled by the geometry of the immeasurably small versus the geometry of immeasurably large. Measurements in the former case will become less and less precise but not in the latter case. When we extend constructions in space to the immeasurably large, an association has to be made between the unlimited and the infinite; the first applies to relations of a topological nature, the second to those of a metric nature.

From the discussions of Riemann at the beginning of the development of modern geometry, we see the importance of the relationship between metric geometry and topology. This, indeed, became a, if not the, central theme in the development of geometry in the twentieth century.

2.2. **Geometry based on the study of a family of linear subspaces.** In 1865, Julius Plücker (1801-1868) studied line geometry, which focused on the space of projective lines in a three dimensional projective space. He introduced Plücker coordinates. This was soon generalized by Hermann Graßmann (1809-1877) to study the space of all linear subspaces of a fixed vector space. This space was later called Grassmannian, which is a universal space for the study of bundles over a manifold. The global topology of Grassmanians played a fundamental role in differential topology.

In 1879, Hermann Schubert (1848-1911) introduced a cell structure on the Grassmanian spaces, which gave the basic homology structures of the Grassmanian manifold (where $Gr(k, n)$ is, by definition, the space of all linear subspaces of a fixed dimension k in a fixed linear space of dimension n.) The structure of their intersections gives rise to the product structures of the homology. The important concept of exterior algebra was introduced by Graßmann in 1844. But it was largely ignored until Poincaré (1854-1912) and Cartan (1869-1951) introduced the concept of differential forms and its calculus with exterior differentiation.

In 1928, Cartan suggested that the differential forms should be linked to the topology of the manifold, and he conjectured that the cohomology defined by differential forms is isomorphic to simplicial cohomology by pairing them together and then integrating differential forms over splicing chains [9]. This fundamental theorem was proved by his student Georges de Rham (1903-1990) in his 1931 thesis [35].

In the 1930s W.V.D. Hodge (1903-1975) discovered the star operator acting on forms, which can be used to define the concept of duality in de Rham theory. Hodge then generalized the work of Hermann Weyl (1885-1955) in 1913 [55] on Riemann surfaces to higher dimensional manifolds using the star operator. He also found the (p, q) decomposition of the cohomology for algebraic manifolds, which ties into his famous conjecture, which held that algebraic cycles represent exactly those (p, p) classes of the manifold [42–44]. This is perhaps the most important conjecture in algebraic geometry that is still unsolved. The periods of (p, q) forms over integer cycles provide the analytic information regarding the Hodge decomposition, which led to Torelli type theorems. The study of periods is a deep subject, dating back to Euler, Gauss, and others.

2.3. Symmetries in geometry.

Motivated by the works of Niels H. Abel (1802-1829) and Evariste Galois (1811-1832) in group theory, Sophus Lie (1842-1899) on contact transformations, Felix Klein (1849-1925), Wilhelm Killing (1847-1923), and others, the notion of Lie groups emerged as an important topic in the late 1860's.

In 1872, Klein initiated the Erlangen program of clarifying geometry based on the continuous group of global symmetries [46]. Examples include: **(a)** projective geometry, **(b)** affine geometry, and **(c)** Möbius geometry.

2.3.1. Projective geometry.

Projective geometry is one of the most classical and influential subjects in geometry. The group of projective collineations is the most encompassing group, which can transform "points at infinity" to finite points. This subject delves into geometric properties that are invariant of such transformations, including incidence relations between linear subspaces and the important concept of duality that came out of such considerations. These concepts form the foundation of modern developments in topology, geometry and algebraic geometry. Major important contributors include:

- Papas of Alexandria (third century)
- Gerard Desargues (1591-1661)
- Blaise Pascal (1623-1662)

- Joseph Diez Gergonne (1771-1859)
- Jean Victor Poncelet (1788-1867)
- August Ferdinand Möbius (1790-1868)
- Jakob Steiner (1796-1863)

The subject of projective geometry was gradually pursued in two different directions: One is the rich theory of algebraic curves developed by Abel, Riemann, Max Noether (1844-1921), and others. Classical projective geometry and invariant theory were used extensively. Italian algebraic geometers, including Gino Fano (1871-1952), Federico Enriques (1871-1946), Benimano Segre, (1903-1971), and Francesco Severi (1879-1961) extended the subject of algebraic curves to algebraic surfaces and some special higher dimensional varieties.

In the other direction, projective differential geometry developed as a mixture of two approaches-one from Riemannian geometry, where one studies local invariants, and the other from the Erlangen program for characterizing geometries according to their symmetries. Contributors include:

- Ernest Julius Wilczynski, (1876-1932)
- Eduard Čech, (1893-1960)
- Wilhelm Blaschke (1885-1962)

At the beginning of twentieth century, many Japanese and Chinese geometers studied the subject of projective differential geometry. This includes Shiing-Shen Chern–the main subject of this paper, of course–and Bu-Chin Su (1902-2003).

2.3.2. **Möbius geometry.** Möbius geometry, also called conformal geometry, relates to manifold properties invariant under the conformal group. It is a special case of the Lie sphere geometry. The subject is very powerful in two dimensions and led to the study of discrete subgroups of the conformal group and conformally flat manifolds of higher dimension. Joseph Liouville (1802-1889) and Poincaré studied the equation that transforms a metric conformally to one with constant scalar curvature. Weyl identified the Weyl tensor, which is part of the curvature tensor responsible for conformal changes of the metric. In four dimensional manifolds, the Weyl tensor can be decomposed further to self dual and anti-self dual parts.

2.3.3. **Affine geometry.** Affine geometry, which was pursued by Guido Fubini (1879-1943), Wilhelm Blaschke, and Eugenio Calabi (1923-) involves the study of differential invariants of hypersurfaces that are invariant under affine transformations of the ambient linear space. The invariants of the affine transformation group have offered a valuable tool for solving Monge–Ampère equations. The important question of classifying (convex) affine spheres gave rise to elliptic Monge–Ampère equations of different types. There is also a concept of affine minimal surfaces, and Chern suggested the problem of finding a Bernstein-type theorem for them. That problem was solved in 2005 by Xu Jia Wang (1963-) and Neil Trudinger (1942-).

3. THE BIRTH OF MODERN DIFFERENTIAL GEOMETRY

André Weil and Hermann Weyl were two giants of twentieth century mathematics. Weil said:[2] "The psychological aspects of true geometric intuition will perhaps never be cleared up...Whatever the truth of the matter, mathematics in our century would not have made such impressive progress without the geometric sense of Cartan, Hopf, Chern, and a very few more. It seems safe to predict that such men will always be needed if mathematics is to go on as before."

Besides the above mentioned great geometers, we should also pay tribute to the outstanding contributions of Levi-Civita, Weyl, Weil, Whitney, Morse, and Hodge. Levi-Civita was the first (in 1917) to introduce the concept of parallel transport in Riemannian geometry. The importance of this concept should not be underestimated. For it provides a way of comparing the information observed at different points of the manifold by transporting the result obtained at one point to another point along a path joining the two points. But unfortunately, the result of this comparison depends on the choice of the path. This leads to an ambiguity that is rather similar to the gauge transformation that Weyl later proposed for gauge theory as part of his attempt to understand gravity coupled with electromagnetism. In 1918, Weyl proposed that his gauge group would be the multiplicative group of positive real numbers, and he also applied conformal change to the metric. In this earlier theory, parallel transportation does not preserve the length, which is something that Einstein objected to.

In 1928, following the work of Fock and London in quantum mechanics, Weyl realized that the correct gauge group is the circle group–the multiplicative group of complex numbers with norm one. Weyl proposed that while the equivalence principle dictates the laws of gravity, the gauge principle dictates the laws of matter. The simplest and most natural action principle in general relativity is the Hilbert action, which is the integral of scalar curvature. And the simplest and most natural action principle in gauge theory is the Weyl action integral, which is the square integral of the curvature tensor.

3.1. **The birth of modern differential geometry.** In the beginning of the last century, Cartan combined Lie group theory and the invariant theory of differential systems to develop the concept of generalized spaces, which includes both Klein's homogeneous spaces and Riemann's local geometry. In modern terminology, he introduced the concept of principle bundles and a connection in a fiber bundle. This is so-called non-abelian gauge theory. It generalized Levi-Civita's theory of parallelism. According to that theory, we have a fiber bundle $\pi\colon E \to M$, whose fibers $\pi^{-1}(x)$, $x \in M$, are homogeneous spaces acted on by a Lie group G. A connection is an infinitesimal transport of the fibers compatible with the group action by G.

While Graßmann introduced exterior forms, Cartan and Poincaré introduced the operation of exterior differentiation. Cartan's theory of a Pfaffian system and his theory of prolongation created invariants for solving the equivalence problem in geometry.

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[2]See [22].

Cartan's view of building invariants by moving frames had a profound influence on Chern.

Heinz Hopf (1894-1971) and Poincaré initiated the study of differential topology by proving that the sum of the indices of a vector field on a manifold can be used to calculate the Euler number of the manifold. Hopf tackled the hypersurface case of Gauss–Bonnet in his 1925 thesis. In 1932, Hopf emphasized that the integrand can be written as a polynomial of components of the Riemann curvature tensor. In 1935, Hopf's student Eduard Stiefel (1909-1978) generalized this work on vector fields to multi-vector fields of the tangent bundle and defined the Stiefel–Whitney classes for tangent bundles. At around the same time, Hassler Whitney (1907-1989) obtained the same characteristic class for a general sphere bundle.

3.2. **Chern: the father of global intrinsic geometry.** The work of Hopf had a deep influence on Chern's later work. As Chern said: "Riemannian geometry and its generalization in differential geometry are local in character. It seems a mystery to me that we do need a whole space to piece the neighborhood together. This is achieved by topology."

Both Cartan and Chern saw the importance of fiber bundles for problems in differential geometry. In 1934, Charles Ehresmann (1905-1979), a student of Cartan, wrote a thesis on the cell structure of complex Grassmanians and showed that its cohomology has no torsion. This paper had a strong influence on Chern's later paper on Chern classes. Ehresmann went on to formulate the concept of connections in the more modern terminology initiated by Cartan.

4. Shiing-Shen Chern: a great geometer

4.1. **Chern's education.**

4.1.1. **Chern's elementary and undergraduate education.** Shiing-Shen Chern was born on Oct. 26, 1911 in Jiaxing, China and died on Dec. 3, 2004 in Tianjin. He studied at home for his elementary education and spent four years in high school. At age fifteen, he entered Nankai University and then spent another four years (1930-1934) at Tsinghua University. In his undergraduate days, he studied:

- Coolidge's non-Euclidean geometry: geometry of the circle and sphere.
- Salmon's book: Conic sections and analytic geometry of three dimensions.
- Castelnuovo's book: Analytic and projective geometry.
- Otto Stande's book: Fadenkonstruktionen.

His teacher, Professor Dan Sun, studied projective differential geometry, a subject pioneered by E.J. Wilczynski in 1901 and followed by Fubini and Čech. Chern's master thesis was on projective line geometry, which concerns hypersurfaces in the space of all lines in three-dimensional projective space. Chern studied line congruences–the two-dimensional submanifold of lines and their oscillation by quadratic line complexes. At the end of his undergraduate years, he wrote four papers on projective differential geometry.

4.1.2. **Chern's education with Blaschke.** In 1932 Blaschke visited Peking, where he lectured on topological questions in differential geometry. He discussed the pseudo-group of diffeomorphisim and its local invariants. Blaschke's lecture made an impression on Chern who started to think about global differential geometry and came to recognize the importance of algebraic topology. He read Veblen's book, Analysis Situs (1922).

The chairman of the math dept at Tsinghua University was Professor Cheng who later became Chern's father in law. He helped Chern get a fellowship, which Chern used to go to the University of Hamburg in 1934 to study under Blaschke. Chern wrote a doctoral thesis on web geometry that was supervised by Blaschke. Emil Artin (1898-1962), Erich Hecke (1887-1947) and Erich Kähler (1906-2000) were also in Hamburg. Blaschke worked on web geometry and integral geometry at that time.

Chern started to read Seifert–Threlfall (1934) and Alexandroff–Hopf (1935). Chern also began to learn integral geometry, which can be said to have been initiated from the formula of Morgan Crofton (1826-1915) on calculating the length of a plane curve by determining the probability of throwing a needle of fixed length onto the plane in such a way that it would exactly intersect this curve. The other founder of integral geometry was Johann Radon (1887-1956) who invented the Radon transform, which is now used extensively in medical imaging. (The basic idea is to reconstruct a geometric figure by slicing the figure with moving planes.) Chern was very fond of integral geometry, partly because of the tradition in Hamburg created by Radon that inspired Blaschke and his group of students. (Radon was a professor from 1919 to 1922 in the newly formed University of Hamburg.) Both Chern and Luis Santaló (1911-2001) were students of Blaschke around the same time. Santaló became a major leader in integral geometry after Blaschke. Chern wrote several papers on the subject, with the first appearing in 1939 [12].

4.1.3. **Chern's education with Kähler.** In Hamburg, Erich Kähler (1906-2000) lectured on Cartan–Kähler theory, as described in Kähler's book, *Einführung in die Theorie der Systeme von Differentialeichungen*. In 1933, Kähler published the first paper [45] in which Kähler geometry was unveiled. It is a remarkable paper that introduced several important concepts. For example, Kähler computed the Ricci tensor of a Kähler metric to be the complex Hessian of the log of the volume form. He observed that the condition of the metric was of the Kähler–Einstein type. That result came from the solution of a complex Monge–Ampère equation, which turned out to be a source of many examples for him. He also proved that the Ricci form defines a closed form, which gives rise to a de Rham cohomology class that is independent of the choice of the Kähler metric.

This is the first Chern form of the Kähler manifold. Chern certainly was influenced by this paper, as he was a student in Hamburg where Kähler's work was carried out. In the last thirty years of his life, Chern told many students that he would like to spend his time teaching them the powerful concept of moving frames invented by Cartan. He probably learned Cartan–Kähler theory from Kähler in Hamburg in 1934 when he was taking a class from him and ended up being the only student in that class.

When Chern graduated, he earned a postdoctoral fellowship in 1936 to pursue further studies in Europe. Blaschke advised him either to stay in Hamburg to study with Artin or go to Paris to study with Cartan. He took the latter choice. From 1936 to 1937, Chern was in Paris, learning from Cartan about moving frames (principle bundles, in modern terminology), the method of equivalence, and more on Cartan–Kähler theory. He spent ten months in Paris and met Cartan every two weeks. Chern went back to China in the summer of 1937. He spent a few years studying Cartan's work. He said that Cartan wrote more than six thousand pages in his life, and Chern read at least seventy to eighty percent of these works. He read some of these papers over and over again. With World War II soon approaching, Chern found time to read and think in isolation [26].

Chern's comment on Cartan. Chern commented on the influence Cartan had on him in the following way [26]:

- "Undoubtedly one of the greatest mathematician of this century, his career was characterized by a rare harmony of genius and modesty."
- "In 1940, I was struggling to learn Cartan. I realized the central role to be played by the notion of a connection and wrote several papers associating a connection to a given geometrical structures."

Weyl's comment on Cartan. Weyl who studied with Hilbert, was one of the great mathematicians of all time. He had this to stay about Cartan: "Cartan is undoubtedly the greatest living master in differential geometry. Nevertheless, I must admit that I found the book, like most of Cartan's papers, hard reading." It was Cartan, around 1901, who first formulated many local geometric problems as generalizations of the Pfaff problem (which described the Lagrangian submanifolds associated with a contact 1-form). Cartan proposed that one should consider, instead of a single 1-form, a collection of 1-forms on a manifold M, and then determine the conditions for finding the maximal submanifolds N of M to which all of the 1-forms in $\mathcal{I}$ pullback are zero. He found sufficient conditions for this but had to use the Cauchy–Kovalewski theorem to solve a sequence of initial value problems in order to construct the maximal submanifolds. So his theory was only valid in the real-analytic category (which did not bother many people at the time.)

4.2. **The equivalence problem.** In modern terms, we would say that Cartan formulated his answer in terms of the algebra of the differential ideal on M generated by the collection of 1-forms in $\mathcal{I}$. Cartan's version of this result sufficed for nearly all of Cartan's applications. In 1933, Kähler found that Cartan's theory could be naturally generalized to the case of a differential ideal on M that was generated by forms of arbitrary degree (not just 1-forms), and he reformulated Cartan's "Test for Involutivity" to cover this more general case. That later became known as the Cartan–Kähler Theorem.

The tools of Cartan–Kähler theory held a powerful sway over Chern, and he learned them well. His skill in constructing forms for the Gauss–Bonnet theorem and the characteristic forms have not been surpassed by any geometer that I know of.

It is also instructive to learn the history of non-abelian gauge theory, which involves connections over vector bundles or principal bundles. In the beginning of the 20th century, Cartan recognized right away that the work of Levi-Civita and Jan Arnoldus Schouten (1883-1971) could be generalized to cover "covariant differentiation" of many different kinds of tensor fields on manifolds endowed with geometric structures. In fact, he had already worked out, in his concept of equivalence, a general method for computing curvature invariants and canonical parallelizations of what we now recognize as principal ever since his famous papers on pseudo-groups came out in the early 1900s.

Throughout the early 1920s, Cartan published papers about intrinsic "connections" on manifolds endowed with (pseudo-)Riemannian, conformal, or projective structures, as well as many others (which he called "generalized spaces"). In his 1926 book on Riemannian geometry, he talked about the covariant differentiation of tensor fields.

Of course, when Chern published the theory of Chern forms in 1946, he was well versed in the notion of unitary connections on bundles. By 1950, both Ehresmann and Chern had written detailed survey papers about connections over general bundles. Chern summarized the works of Cartan and himself on connections and characteristic forms for general vector bundles in modern language in his plenary talk before the International Congress of Mathematicians held at Harvard in 1950 [17]. In fact, when Chern left China in late 1948 and arrived in Princeton in early 1949, he gave a series of lectures at the Institute for Advanced Study's Veblen Seminar. The lectures were written up in 1951 when Chern was in Chicago. The title of his talk was called "Topics in Differential Geometry" [16]. In this lecture notes, Chern explained again the theory of connections.

The subject is now called non-abelian gauge theory by physicists. The abelian version was founded and pioneered by Weyl in 1928. In fact, Weyl coined the term gauge principle to explain the fundamental structure of matter. In his famous book, *Gruppentheorie und Quantenmechanik*, Weyl observed that the electromagnetic equation and the relativistic Schrödinger equation are invariant under the gauge transformation if one takes the gauge group to be the circle. This was observed by London, who noted that the previous gauge group of positive real numbers, advanced by Weyl in 1918, is not compatible with the wave function in quantum mechanics.

As Weyl stated: "This 'Principe of gauge invariance' is quite analogous to that previously set up by the author [Weyl himself], on speculative ground, in order to arrive at a unified theory of gravitation and electricity. But I now believe that this gauge invariance does not tie together electricity and gravitation, but rather electricity and matter in the manner described above. This new principle of gauge invariance has the character of general relativity since it contains an arbitrary function and can certainly only be understood with reference to it."

Weyl also said: "If our view is correct, then the electromagnetic field is a necessary accomplishment of the matter wave field and not gravitation." Weyl proposed in 1918 that the action was the integral of the square of the curvature. (In 1918, Weyl was proposing a conformal change of the metric, coupling the conformal factor with the gauge group.) In fact, in a gauge theory, the simplest gauge invariant scalar quantity is the integral of the square of its curvature.

In 1954, C.-N. Yang (1922-) and Robert Mills (1927-1999) applied this theory to explain isospin in particle physics. But since they did not know how to quantize the theory, they did not know how to compute the mass, as was pointed out by Wolfgang Pauli (1900-1958) who had also developed the non-abelian version of Weyl's gauge theory. Apparently, neither Pauli, Yang, nor Mills knew the works of Cartan, Chern. and others despite the fact that Yang was a student at the University of Chicago and a postdoctoral fellow in Princeton at the same time that Chern was at these institutions. Yang's father also happened to be a teacher of Chern.

4.3. **Chern's work in geometry.** Let us now explain in greater detail Chern's works in geometry. We shall divide the discussion of this work into four stages:

1. From 1932 to 1943: Tsinghua University, Hamburg, Paris, and Kunming.
2. From 1944 to 1949: Princeton, Nanjing, and Princeton.
3. From 1950 to 1960: The Chicago days.
4. From 1960 to 2004: Berkeley and Nankai University.

Much of Chern's work is related to the problem of equivalence, which dated back to Riemann. In 1869, Christoffel and Lipschitz solved a special form of the equivalence problem in Riemannian geometry. It was also called the form problem:

Problem 1. *To decided when two* ds^2 *differ by a change of coordinate, Christoffel introduced the covariant differentiation now called Levi-Civita connection.*

It was Cartan who formulated a more general form of the equivalence problem, which can be stated as follows:

Problem 2. *Given two sets of linear differential forms* θ^i, θ^{*j} *in the coordinates* x^k, x^{*l} *respectively, where* $1 \leq i, j, k, l \leq n$ *are both linearly independent. Given a Lie group* $G \subset \mathrm{GL}(n, \mathbb{R})$, *find the conditions that there are functions*

$$x^{*l} = x^{*l}(x^1, \ldots, x^n)$$

such that θ^{*j}, *after the substitution of these functions, differ from* θ^i *by a transformation of* G.

The problem generally involves local invariants, and Cartan provided a procedure to generate such invariants.

4.3.1. **Chern (1932-1943).** Chern continued the tradition of Cartan and applied the Cartan–Kähler theory to solve various geometric questions related to the equivalence problem. For example, in projective differential geometry, he was interested in the following question:

Problem 3. *Find a complete system of local invariants of a submanifold under the projective group and interpret them geometrically through osculation by simple geometrical figures.*

Chern studied web geometry, projective line geometry, invariants of contact pairs of submanifolds in projective space, and transformations of surfaces (related to the Bäcklund transform in soliton theory). Another typical problem in projective differential geometry is to study the geometry of path structure by normal projective connections. Tresse (a student of Sophus Lie) studied paths defined by integral curves of $y'' = F(x, y, y)$ by normal projective connections in space (x, y, y'). Chern generalized this to n- dimensions: Given $2(n-1)$-dimensional family of curves satisfying a differential system such that through any point and tangent to any direction at the point, there is exactly one such curve. Chern defined a normal projective connection and then extended it to families of submanifolds.

The first major work that Chern did was in 1939 when he studied integral geometry that was formerly developed by Crofton and Blaschke. Chern observed that this theory could be best understood in terms of two homogeneous spaces with the same Lie group G. Hence there are two subgroups, H and K

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\begin{array}{ccc}
& G & \\
\swarrow & & \searrow \\
G/H & & G/K.
\end{array}
$$

Two cosets aH and bK are incident to each other if they intersect in G. Important geometric quantities in G/H can be pulled back to G and, depending on the properties of G, we can then twist the pulled back quantity by some other quantities defined on G. By employing this procedure, we can obtain important geometric invariants for G/K.

This work preceded the important contributions of the Russian school led by Israel Gelfand (1913-2009) and Shigeru Mukai (1953-). The transformation defined in this way is sometimes called the Fourier–Mukai transformation.

In his work in integral geometry, Chern generalized several important formulas of Crofton and, much later, he used this setting to generalize the kinematic formula of Poincaré, Santaló, and Blaschke. Weil commented: "This work of Chern ... lifted the whole subject at one stroke to a higher plane than where Blaschke's school had left it. I was impressed by the unusual talent and depth of understanding that shone through it."

4.3.2. **Chern's visit to Princeton (1943).** In 1943, Chern went from Kunming to Princeton, invited by Oswald Veblen (1880-1960) and Weyl. This visit took place, of course, while World War II was still underway. It took Chern seven days to fly by military aircraft from Kumming to Miami via India, Africa, and South America. He arrived at Princeton in August by train. (It took him five years before he was able to see his newborn son again.) Weyl was Chern's hero. But it was Weil who encouraged Chern to look into the fiber bundle theory of Cartan and Whitney. Weil also suggested

that he read two key papers by Todd and Eger (cited below), who defined certain 'canonical classes' in algebraic geometry–unlike Stiefel–Whitney classes which were only defined mod two [22]. However, these works did rest on some unproved assumptions. (Note that the classes defined by Todd and Eger were shown to be characteristic classes, as defined by Chern in the late 1940s and by Hodge.)

- John Arthur Todd (1908-1994) published his work on Todd class: "The geometric invariants of algebraic loci" in 1937 in *Proceedings of the London Math Society.*
- Max Eger published his work, "Sur les systems canoniques d'une varietie algebrique a plusieurs dimensions" in 1943 in *Annales Scientifique de l'Ecole Normale Supérieure.*

Gauss–Bonnet formula. Chern told everyone that his best work was his intrinsic proof of the Gauss–Bonnet formula. Here is a brief history of that formula:

- Carl Friedrich Gauss proved the formula in the case of the geodesic triangle in 1827 ("Disquistiones Circa superficies Curvas"). He considered a surface in $\mathbb{R}^3$ and used a Gauss map.
- In 1948, Pierre Ossian Bonnet (1819-1892) generalized this to any simply connected domain bounded by an arbitrary curve in "Mémoire sur la théorie générale des surfaces."
- Walther von Dyck (1856-1943) went further in 1888, generalizing this relation to arbitrary genus ("Beiträge zur analysis situs").
- Hopf generalized the formula to codimension one hypersurfaces in $\mathbb{R}^n$.
- Carl B. Allendoerfer (1911-1974) in 1940 and Werner Fenchel (1905-1988) studied the closed orientable Riemannian manifold, which can be embedded in Euclidean space.
- Allendoerfer and Weil in 1943 extended the formula to the closed Riemannian polyhedron and hence to the general closed Riemannian manifold in "The Gauss–Bonnet theorem for Riemannian polyhedra," *Amer. Math. Soc.*, 53(1943), 101–129.

Weil made these comments in his introduction to the *Selected Papers of S S Chern*:

- Following in the footsteps of Weyl and other writers, the latter proof, resting on the consideration of "tubes," did depend (although this was not apparent at that time) on the construction of a sphere bundle, but of a non-intrinsic one, viz. the transversal bundle for a given immersion.
- Chern's proof operated explicitly for the first time with an intrinsic bundle, the bundle of tangent vectors of length one, thus clarifying the whole subject once and for all.

Let us explain what Chern did: In the simplest two dimensional case, he wrote, in terms of a moving frame, the structure equation for a surface is

$$d\omega_1 = \omega_{12} \wedge \omega_2$$
$$d\omega_2 = \omega_1 \wedge \omega_{12}$$
$$d\omega_{12} = -K\omega_1 \wedge \omega_2$$

where ω_{12} is the connection form and K is the Gauss curvature. The connection form here becomes the transgression form for the curvature form, and it is defined only on the unit tangent bundle. This was generalized to higher dimensional manifolds by Chern by the much more complicated procedure of putting combinations of products of connections and curvature forms together.

If the unit vector e_1 is given by a globally defined vector field V by

$$e_1 = \frac{V}{\|V\|}$$

at points where $V \neq 0$, then we can apply Stokes' formula to obtain

$$(1) \qquad -\int_M K\omega_1 \wedge \omega_2 = \sum_i \int_{\partial B(x_i)} \omega_{12}$$

where $B(x_i)$ is a small disk around x_i with $V(x_i) = 0$. Each term in the right hand side of (1) can be computed via the index of the vector field of V at x_i. According to the theorem of Hopf and Poincaré, summation of indices of a vector field is the Euler number. This proof of Chern is new even in two dimensions. In his higher-dimensional proof, the bundle is the unit tangent sphere bundle. The curvature form Ω_{ij} is skew-symmetric. The Pfaffian is

$$\text{Pf} = \sum \varepsilon_{i_1,\dots,i_{2n}} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2n-1} i_{2n}}.$$

The Gauss–Bonnet formula is

$$(-1)^n \frac{1}{2^{2n}\pi^n n!} \int_M \text{Pf} = \chi_{\text{top}}(M).$$

Chern managed to find, in what was a true tour de force, a canonical form Π on the unit sphere bundle so that $d\Pi$ is the lift of Pf. This beautiful construction is called transgression and played an important role in the topology theory of fiber bundles. This construction is very important. When it was applied to the Pontryagin forms, it gave rise to the Chern–Simons forms, a joint work with Jim Simons (1938-) that was completed twenty or so years later.

Discovery of Chern classes. In the preface to Chern's selected works, Weil said that when Chern arrived in Princeton in 1943, both of them were deeply impressed by the work of Cartan and the masterly presentation by Kähler in the following paper: "Einfuhrung in die Theorie der Systeme von Differentialgleichungen." Both of them

recognized the importance of fiber bundles in geometry. However, Weil did not seem to realize that Chern was also influenced by two works of Lev Pontryagin (1908-1988):

1. "Characteristic cycles on manifolds," C.R.(Doklady) Acad. Sci. USSR, vol. 35 (1942) 34-37.
2. "On some topological invariants of Riemannian manifolds," C.R.(Doklady) Acad. Sci. USSR(N.S.), vol. 43(1944) 91-94.

These two papers were mentioned by Chern in the preface of his paper on Chern classes [14]. In the second paper, Pontryagin introduced closed forms defined by the curvature form. He proved that the de Rham cohomology defined by the closed form is independent of the metric that defines the curvature forms. Pontryagin could only identify his curvature form to be a characteristic form when the manifold can be isometrically embedded into Euclidean space. I believe Chern attempted to solve this problem left in the work of Pontryagin after he succeeded in giving the intrinsic proof of the Gauss–Bonnet formula. He did not succeed in carrying out the calculation for the real Grassmannians, whose cell structure is more complicated, but he was able to do so for the complex Grassmanians.

Chern said: "My introduction to characteristic class was through the Gauss–Bonnet formula, known to every student of surfaces theory. Long before 1943, when I gave an intrinsic proof of the n-dimensional Gauss–Bonnet formula, I knew by using orthonormal frames in surface theory that the classical Gauss–Bonnet is but a global consequence of the Gauss formula, which expresses the 'Theorema Egregium.'"

The algebraic aspect of this proof is the first instance of a construction later known as transgression, which was destined to play a fundamental role in the homology theory of fiber bundles and in other problems as well.

Cartan's work on frame bundles and de Rham's theorem were always in the background of Chern's thoughts. The history of fiber bundles can be briefly described as follows:

- Stiefel in 1936 and Whitney in 1937 introduced Stiefel–Whitney classes. It is only defined in mod two.
- Jacques Feldbau (1914-1945) in 1939, Ehresmann in 1941, 1942, 1943, Chern in 1944, 1945, and Norman Steenrod (1910-1971) in 1944 studied the topology of fiber bundles.
- Pontryagin in 1942 introduced Pontryagin classes. He also associated topological invariants to the curvature of Riemannian manifolds in 1944 (Doklady).

In his proof of the Gauss–Bonnet formula, Chern used one vector field and looked at its set of zero to find the Euler characteristic of the manifold.

If we replace a single vector field by k vector fields $s_1, \ldots, s_k$ in general position, they are linearly independent from a $(k-1)$-dimensional cycle whose homology class is independent of the choice of s_i. This was done by Stiefel in his 1936 thesis.

Chern considered a similar procedure for complex vector bundles. In the proof of the Gauss–Bonnet formula, he used curvature forms to represent the Euler class by the zero set of the vector field. It is therefore natural to do the same for the other Chern classes

using the set of degeneracy for k vector fields. Whitney in 1937 considered sections for more general sphere bundles, beyond tangent bundles, and looked at it from the point of view of obstruction theory. He noticed the importance of the universal bundle over the Grassmannian $\mathrm{Gr}(q, N)$ of q planes in $\mathbb{R}^N$. In 1937, Whitney showed that any rank q bundle over the manifold M can be induced by a map $f \colon M \to \mathrm{Gr}(q, N)$ from this bundle. When N is large, Pontryagin in 1942 and Steenrod in 1944 observed that the map f is unique up to homotopy. The characteristic classes of the bundle is given by

$$f^* \mathrm{H}^\bullet(\mathrm{Gr}(q, N)) \subset \mathrm{H}^\bullet(M).$$

The cohomology $\mathrm{H}^\bullet(\mathrm{Gr}(q, N))$ was studied by Ehresmann in 1936, and it is generated by Schubert cells. In a recollection of his own works in the 1990s [14], Chern said: "It was a trivial observation and a stroke of luck, when I saw in 1944 that the situation for complex vector bundles is far simpler, because most of the classical complex spaces, such as the classical complex Grassmann manifolds, the complex Stiefel manifolds, etc., have no torsion."

For a complex vector bundle E, Chern defined Chern classes in three different ways: by obstruction theory, by Schubert cells, and by curvature forms of a connection on the bundle. He proved their equivalence.

Although the theory of Chern classes has had a much bigger impact on mathematics than his proof of the Gauss–Bonnet theorem, Chern considered the latter proof to be his best work. The formula was in fact carved into his tombstone at Nankai University. I believe the reason he ranked things this way is that he got the idea for Chern classes from his proof of the Gauss–Bonnet theorem. Also, in his proof of the Gauss–Bonnet formula, he started to appreciate the power of studying the geometry of forms on the intrinsic sphere bundle of tangent vectors with length one. This approach, based on obstruction theory, is parallel to the way that Stiefel generalized Hopf's vector field theory to Stiefel–Whitney classes by looking at them as an obstruction to multivector fields that are linearly independent.

As for the curvature forms, the representations of Chern classes by curvature forms offer a clear analogue to the Gauss–Bonnet formula. Therefore, Chern worked out the Chern form for unitary connections. When Weil reported Chern's work in the Bourbaki Seminar, he formulated it so that it applies to connections based on connections with any compact Lie group [22].

According to Chern himself, he knew the formula for general G-connections. But he was not aware of the proof that the cohomology classes were independent of the choice of connections. In a way, this is surprising because Weil simply forms a family of connections by linearly joining two connections together. One can then differentiate the characteristic forms defined by the connections in this family to obtain its transgression form.

This kind of idea was used by Kähler in 1933 to prove that the first Chern class, as represented by the Ricci curvature form, is independent of the Kähler metric. The same idea was also used by Pontryagin to prove a similar statement for Pontryagin classes.

In 1945, Chern was invited to give a plenary address in the summer meeting of the American Mathematical Society. His report appeared in a 1946 issue of the *Bulletin of the American Mathematical Society* (Vol. 52). It is entitled: "Some new viewpoints in differential geometry in the large." In his review of this paper, Hopf wrote that Chern's work had ushered in a new era in global differential geometry.

Chern returned to China in April of 1946 where he became the deputy director of the Institute of Mathematics for Academia Sinica in Nanking. In this period, and also in the period when he was teaching at Tsinghua University (which temporarily merged with other universities during the Second Sino-Japanese War to form Southwestern University in Kunming), he trained a few young Chinese mathematicians that became influential in China. The most notable mathematicians were Hsien-Chung Wang (1918-1978), Kuo-Tsai Chen (1923-1987), and Wen-Tsun Wu (1919-2017). Although none of them were PhD students of Chern, their contributions to topology and geometry had a deep influence on other Chinese differential geometers.

When Friedrich Hirzebruch (1927-2012) was writing his paper, "Transferring some theorems of algebraic surfaces to complex manifolds of two complex dimensions" ("Über-tragung einiger Sätze aus der Theorie der algebraischen Flächen auf komplexe Man-nigfaltigkeiten von zwei komplexen Dimensionen." J. Reine Angew. Math. 191 (1953), 110–124), he noticed that some of the results of that paper could have been generalized to higher dimensions.[3] But the so-called duality formula had not yet been proved. This formula says that the total Chern class of the direct sum of two complex vector bundles equals the product of the total Chern classes of the summands. Hirzebruch then wrote that during his proofreading of his paper, both Chern and Kunihiko Kodaira (1915-1997) informed him that the duality formula would be proven in a forthcoming paper by Chern: "On the characteristic classes of complex sphere bundles and algebraic varieties" (*Amer. J. Math.* 75(1953), 565–597).

Looking back on his career, Hirzebruch said: "My two years (1952–1954) at the Institute for Advanced Study were formative for my mathematical career. I had to study and develop fundamental properties of Chern classes, introduced the Chern character, which later (in joint work with Michael Atiyah [1929-2019]) became a functor from K-theory to rational cohomology."

The developments led by Kodaria and Hirzebruch set the stage for the modern era of algebraic geometry. In this development, Chern classes were utilized in the most essential way. I believe that Chern was amazed at seeing such powerful uses of his classes within ten years of the publication of his paper. Chern said that in 1948, Weil told several people, including C.-N. Yang, that the integrity of Chern classes would be used to quantize physical systems. Chern might have thought it was a joke at the time. But this turned out to be a trenchant observation, much more so than most people

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[3]Hirzebruch wrote to me his correspondence with Todd. When he proved the formula to express arithmetic genus in terms of Todd classes, Todd wrote to him that in 1939, he has gotten the formula of Hirzebruch up to 12 dimensional algebraic manifolds. Note that higher Chern classes were used in such formula.

could imagine at that time. The same thing happened in the modern development of the secondary Chern classes where the Chern–Simons invariant is the major class. It has been used extensively in theoretical physics as a means to understand anomalies.

The fundamental paper of Chern (1946). In the paper, "Characteristic classes of Hermitian manifolds," Chern also laid the foundation for performing Hermitian geometry on complex manifolds. The concept of Hermitian connections (later called Chern connections) was introduced. If Ω is the curvature form of the vector bundle, one defines

$$\det\left(\mathrm{I} + \frac{\sqrt{-1}}{2\pi}\Omega\right) = \mathbf{1} + c_1(\Omega) + \cdots + c_q(\Omega).$$

The advantage of defining Chern classes by differential forms have tremendous importance in geometry and in modern physics.

The advantage of defining Chern classes by differential forms has been of tremendous importance in geometry and modern physics. An example is the concept of transgression created by Chern. Let ω be the connection form defined on the frame bundle associated to the vector bundle. Then the curvature form is computed via $\Omega = \mathrm{d}\omega - \omega \wedge \omega$ and hence

$$c_1(\Omega) = \frac{\sqrt{-1}}{2\pi}\mathrm{Tr}(\Omega) = \frac{\sqrt{-1}}{2\pi}\mathrm{d}(\mathrm{Tr}(\omega)).$$

Similarly, we have

$$\mathrm{Tr}(\Omega \wedge \Omega) = \mathrm{d}\left(\mathrm{Tr}(\omega \wedge \omega) + \frac{1}{3}\mathrm{Tr}(\omega \wedge \omega \wedge \omega)\right)$$
$$= \mathrm{d}(CS(\omega)).$$

This term $CS(\omega)$ is called the Chern–Simons form and has played a crucial role in three dimensional manifolds, in anomaly cancellation, in string theory, and in condensed matter physics. The physicist Edward Witten–the originator of M-theory in physics and a Fields Medal winner–has acknowledged that his work was strongly influenced by the Chern–Simons function.

The idea of doing transgression on the form level also gives rise to a secondary operation on homology, i.e., a Massey product. It appeared in K.-T. Chen's work on iterated integral. When the manifold is a complex manifold, we can write $\mathrm{d} = \partial + \bar{\partial}$.

In a fundamental paper, Raoul Bott (1923-2005) and Chern (1965) found: for each i there is a canonically constructed $(i-1, i-1)$-form $\tilde{T}c_i(\Omega)$ so that $c_i(\Omega) = \bar{\partial}\partial(\tilde{T}c_i(\Omega))$.

Chern made use of this theorem to generalize Nevanlinna theory of value distribution to holomorphic maps between higher dimensional complex manifolds. The form $\tilde{T}c_i(\Omega)$ plays a fundamental role in Arakelov theory. Simon Donaldson (1957-) used the case $i = 2$ to prove the Donaldson–Uhlenbeck–Yau theorem for the existence of hermitian Yang–Mills equations on algebraic surfaces. For $i = 1$,

$$c_1 = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log \det(h_{\alpha\bar{\beta}})$$

where $h_{\alpha\bar{\beta}}$ is the hermitian metric. The right hand side is the Ricci tensor of the metric.

The simplicity of the first Chern form motivated the Calabi conjecture. Indeed, the simplicity and beauty of geometry over complex number cannot be exaggerated. After his fundamental paper on Chern classes in 1946, Chern explored more details on the multiplicative structure of the characteristic classes.

4.3.3. **Chern (IAS in 1949 and later in Chicago).** In 1950, Chern and E. Spanier were in Chicago, and they wrote a paper on the homology theory of sphere bundles. There they proved the very powerful Thom isomorphism (which had also been proven independently by Thom in the same year) [33]. In his 1953 paper, "On the characteristic classes of complex sphere bundles and algebraic varieties," Chern showed that by considering an associated bundle with the flag manifold as fibers, the characteristic classes can be defined in terms of line bundles. As a consequence, the dual homology class of a characteristic class of an algebraic manifold contains a representative of algebraic cycle. This paper provides the splitting principle in K-theory and, coupled with the Thom isomorphism, allows one to give the definition of Chern classes on the associated bundle, as was done later by Alexander Grothendieck (1928-2014). Hodge also considered the problem of representing homology classes by algebraic cycles. He considered the above theorem of Chern and was only able to prove it when the manifold is a complete intersection of nonsingular hypersurfaces in a projective space.

Chern's theorem is the first and the only general statement for the "Hodge conjecture." It also offers the first direct link between holomorphic K-theory and algebraic cycles. Chern's ability to create invariants for important geometric structures is unsurpassed by any mathematician that I have ever known or heard of. His works on projective differential geometry, on affine geometry, and on Chern–Moser invariants for pseudo-convex domains demonstrate his strength. The intrinsic norm on the cohomology of complex manifolds that he defined with Harold Levine (1922-2017) and Louis Nirenberg (1925-2020) has not yet been fully exploited. Before he died, a major program for him was to carry out the Cartan–Kähler system for more general geometric situations.

In his review article on a book by André Lichnerowicz (1915-1998) in 1955 called "Théorie globale des connexions et des groupes d'holonomie," Chern explained that the classical works of Cartan pointed to the fact that the group concept is the basic underlying idea behind the work of Levi-Civita and Schouten on the theory of connections. He also tried to clear up some confusion pertaining to Cartan's terminology. Cartan's "tangent space" is a fiber in contemporary parlance, and his space of a moving frame is what is now called a principal fiber bundle. Chern said: "The holonomic group is a very natural notion in the theory of connections. However, recent investigations by Marcel Berger (1927-2016) and Isadore Singer (1924-) have shown that its possibilities are rather limited. Except for homogeneous spaces, it is perhaps not a strong invariant."

Many years ago, Singer told me that he attended Chern's class in Chicago and took notes, which he showed to Warren Ambrose (1914-1995) at MIT. Singer and Ambrose worked together to prove what we now call the Ambrose–Singer theorem, which identifies the Lie algebra of the holonomic group by relating it to curvature tensors. Berger in

France developed this idea further and classified all possible Lie groups that may appear as holonomic groups in Riemannian geometry. (A more conceptual proof was given by Simons later).

Holonomic groups, introduced by Cartan in 1926, give rise to the concept of internal symmetry of the manifold and also confer geometric meaning to what modern physicists called supersymmetry. Kähler manifolds are those whose holonomic group is a unitary group. Calabi–Yau manifolds are those whose holonomic group is a special unitary group. Contrary to what Chern expected, manifolds with special holonomy have been among the most fascinating manifolds in modern geometry. The construction of such manifolds depends on nonlinear analysis, which Chern was not very familiar with.

It may be interesting to note that Chern gave a course on Hodge theory for Kähler manifolds in Chicago using potential theory, building upon the work of Kodaria. But in the late 1960s, Chern wrote a booklet called *Complex Manifolds with Potential Theory*. For some reason, Chern abandoned his interest in this direction in Kähler geometry, which was pioneered by Kodaria who initiated the powerful tool of the vanishing theorem in algebraic geometry and applied it to prove the famous Kodaira embedding theorem. In the late 1950s, Chern became curious about the old classical subject of minimal surfaces. He was attracted by the works of Calabi in the global theory of minimal two spheres in higher-dimensional spheres.

4.3.4. **Chern 1960–2004.** In 1967, Chern and Osserman observed that the Gauss map mapping minimal surfaces in higher-dimensional Euclidean spaces into the Grassmanian of two planes in higher-dimensional Euclidean space is anti-holomorphic. Hence one can apply the theory of holomorphic curves to minimal surfaces theory to reprove the work of Bernstein–Osserman on minimal surfaces. (Note that the Grassmanian of two planes has a natural complex structure.) Motivated by this work, Chern wrote several papers regarding harmonic maps from the two sphere into Grassmanians. Some of these papers were done with his student Jon Wolfson. These methods for constructing such harmonic maps were popular at the time, as was recounted by Karen Uhlenbeck (1942-), Fran Burstall, and John Wood. At the same time, when Chern was studying minimal surfaces, he became interested in holomorphic maps between Kähler manifolds. He generalized the work of Lars Ahlfors (1907-1996) on the Schwarz lemma to higher dimensional complex manifolds. He realized negative curvature gives boundedness for holomorphic maps, as was observed by Ahlfors. This provided motivation for the work of Shoshichi Kobayashi (1932-2012) on hyperbolic manifolds and that of Phillip Griffiths (1938-). Chern's lectures on minimal surfaces at Berkeley influenced the important work of Simons on higher-dimensional minimal subvarieties by making important inroads toward the stability questions regarding minimal cones. That, in turn, solves a part of the Bernstein problem, which afforded a better understanding of the singularity of minimal subvarieties. In particular, Simons made an important contribution towards the Bernstein problem in this theory, thereby providing a better understanding of the singularity of minimal subvarities.

The last, most important work that Chern did in the 1970s was with Simons, now called Chern–Simons invariants, and his work with Jürgen Moser (1928-1999), now called Chern–Moser invariants for strongly pseudoconvex manifolds. This work with Simons was motivated by the idea of transgression, which came up in his proof of the Gauss–Bonnet formula. It has become a cornerstone for subsequent work in theoretical physics and condensed matter physics. His work with Moser continued the unfinished efforts of Cartan on the construction of local invariants of domains invariant under biholomorphic transformations.

The Chern–Simons form has became very important in theoretical physics in the past forty years. We briefly outline its development below:

1. In 1978, Albert Schwarz (1934-) introduced a topological quantum field theory that includes Chern–Simons theory. The paper is called: "The partition function of degenerate quadratic functional and Ray–Singer invariants," *Letters in Mathematical Physics*, Vol. 2, 247-252.
2. In 1981, Roman Jackiw and his student Stephen Templeton studied 3D QED with the Chern–Simons term and, in 1982, non-abelian gauge theory and 3D Einstein gravity.
3. In 1981, Laughlin wrote about quantized Hall conductivity in two dimensions and in 1983 on the fractional quantum Hall effect where low energy can be described by the Chern–Simons term. This was followed by Wilczek, Zee, Polyakov, and others.
4. Witten developed the 3D Chern–Simons as a quantum theory related to the Jones polynomial. This paper of Witten prompted a great deal of investigation of knot theory, including the so-called volume conjecture for three dimensional hyperbolic manifolds. The influence of Chern–Simons theory and its applications in modern condensed matter physics is so vast and overwhelming that there is simply not enough space to report on that here.

After these two important works, Chern was still active, continuing to write several papers on different topics including web geometry, harmonic maps into complex Grassmannians, and Lie sphere geometry. But they are less consequential in comparison to his previously mentioned works. This is probably natural as Chern was then in his 70s-an age when mathematicians, even the most extraordinary ones, rarely make their biggest contributions.

5. Conclusion

When I was a student, Chern told me that he was interested in mathematics because it was fun and was the only thing he knew how to do. He felt that he could master very complicated calculations, as was demonstrated in his proof of the Gauss–Bonnet theorem. Despite his tremendous influence in modern geometry, he said that he did not have a global vision, even though many of his peers assumed, as a matter of course, that his work was guided by such an outlook. Chern just followed his intuition to solve problems that seemed interesting to him and held out the promise of being fun things

to work on. And he always emphasized how important it was to him to be surrounded with friends and colleagues who had brilliant minds.

Chern once said: "The importance of complex numbers in geometry is a mystery to me. It is well-organized and complete." He always regretted the fact that ancient Chinese mathematicians never discovered complex numbers. Chern's everlasting contributions to the field of complex geometry make up for that shortcoming in Chinese mathematics over the preceding two thousand years.

During the last part of his career, Chern tried to promote Finsler Geometry. He wrote a book with David Bao and Zhongmin Shen on the subject. Since there was no concrete example of Finsler geometry to model, they had a hard time developing their theory in great depth. In particular, they were not able to apply their theory to the concrete example of Finsler geometry as it appears in Teichmüller space or in Kobayashi hyperbolic manifolds.

Riemann, in his thesis paper, thought about the possibility of replacing Riemannian metrics defined by quadratic differentials with quartic differentials–presumably to handle the geometry of space that is far apart. It will be interesting to see whether rich geometric tools can be developed based on quartic differentials. One has to solve the equivalence problem, i.e., to find complete invariants to determine whether two quartic differential are equal up to the change of variables. Perhaps one can start by taking the symmetric tensor product of two quadratic forms.

Ironically, while Chern was a great admirer of Riemann, Cartan, Weyl, and Weil, he did not think that highly of Einstein and was slow, in general, in reacting to the ideas that came from theoretical physics. He showed no interest, for example, in the part of geometry related to quantum field theory. Yet the dream of Riemann to understand the space of the extremely small requires, in order to be fulfilled, a thorough understanding of quantum field theory and perhaps a new form of quantum geometry, as well.

Nevertheless, Chern was flexible in his approach and outlook. When I mentioned to him that I was working on the Calabi conjecture, he did not think much of it at first–that is until he realized it could be used to address some of problems he wanted to solve in algebraic geometry. After that, he came to appreciate the power of nonlinear analysis in geometry. This was reflected in the series of international conferences on "Differential Geometry and Differential Equations" that Chern organized after he returned to China.

There is no question that Chern is a great mathematician, and there is no doubt that he will always be remembered in the history of mathematics, especially for his contributions to the theory of fiber bundles and its characteristic classes. The International Astronomical Union named an asteroid after Chern in November 2004, one month before his death. This "main-belt" asteroid, 29552 Chern, will circulate through the solar system indefinitely, just as his ideas and teachings will forever permeate the realm of mathematics.

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[18] ———, *On the characteristic classes of complex sphere bundles and algebraic varieties*, American Journal of Mathematics **75** (1953), 565–597. In the preface of this paper, he said "In a recent paper, Hodge studied the question of identifying, for non singular algebraic varieties over the complex field, the characteristic classes of complex manifolds with the canonical systems introduced by M. Eger and J.A. Todd. He proved that they are identical up to a sign, when the variety is the complete intersections of nonsingular hyper surfaces in a projective space. His method does not seem to extend to a general algebraic variety. ...We shall give in this paper a more direct treatment of the problem, by proving that there is an equivalent definition of the characteristic classes, which is valid for algebraic varieties."

[19] _____ , *Review: A. Lichnerowicz, Théorie globale des connexions et des groupes d'holonomie*, Bulletin of the American Mathematical Society **63** (1957), no. 1, 57–59.

[20] _____ , *Holomorphic mappings of complex manifolds*, L'Enseignement Mathématique. Revue Internationale. 2e Série **7** (1961), 179–187 (1962).

[21] _____ , *On holomorphic mappings of hermitian manifolds of the same dimension*, Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), 1968, pp. 157–170.

[22] _____ , *Selected papers*, Springer-Verlag, New York-Heidelberg, 1978. With a foreword by H. Wu and introductory articles by André Weil and Phillip A. Griffiths. In the preface, Weil said: "I was able to tell Chern about the 'canonical classes' in algebraic geometry, as introduced in the work of Todd and Eger. Their resemblance with the Stiefel–Whitney classes was apparent, while they were free from the defect (if it was one) of being defined modulo 2."; their status, however, was somewhat uncertain, since that work had been done in the spirit of Italian geometry and still rested on some unproved assumptions.

[23] _____ , *Affine minimal hypersurfaces*, Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977), 1979, pp. 17–30.

[24] _____ , *General relativity and differential geometry*, Some strangeness in the proportion: a centennial symposium to celebrate the achievements of Albert Einstein, 1980. In this paper, Chern said: "a definitive treatment of affine connections, together with a generalization to connections with torsion, was given by Élie Cartan in his fundamental paper ' Sur les variétés à connexion affine, et la théorie de la relativité généralisée', published in 1923-24. The paper did not receive the attention it deserved for the simple reason it was ahead of its time. For it is more than a theory of affine connections, its idea can be easily generalized to give a theory of connections in a fiber bundle with any Lie group, for whose treatment the Ricci calculus is no longer adequate. ...Specifically the following contributions could be singled out: 1. the introduction of the equations of structure and the interpretation of the so called Bianchi identities as the result of exterior differentiation of the equations of structure. 2. the recognition of curvature as a tensor valued exterior quadratic differential form."

[25] _____ , *Deformation of surfaces preserving principal curvatures*, Differential geometry and complex analysis, 1985, pp. 155–163.

[26] _____ , *Selected papers. Vol. II*, Springer-Verlag, New York, 1989. Chern said: "The general case of a principle bundle with an arbitrary Lie group as structure group, of which my work above is a special case concerning the unitary group, was carried out by Weil in 1949 in an unpublished manuscript. Part of Weil's result was presented in the Bourbaki seminar. The main conclusion is the so called Weil homomorphism which identifies the characteristic classes (through the curvature forms) with the invariant polynomials under the action of the adjoint group."

[27] _____ , *Selected papers. Vol. III*, Springer-Verlag, New York, 1989.

[28] _____ , *Selected papers. Vol. IV*, Springer-Verlag, New York, 1989.

[29] Shiing-Shen Chern and Claude Chevalley, *Obituary: Elie Cartan and his mathematical work*, Bulletin of the American Mathematical Society **58** (1952), 217–250. In the preface of this paper, the authors mentioned "a generalized space (space generalise) in the sense of Cartan is a space of tangent spaces such that two infinitely near tangent spaces are related by an infinitesimal transformation of a given Lie group. Such a structure is known as a connection. The tangent space may not be the spaces of tangent vectors. This generality, which is absolutely necessary, gave rise to misinterpretation among differential geometers. It is now possible to express these concepts in a more satisfactory way, by making use of the modern notion of fiber bundles." On page 243 of this paper, the authors said "without the notion and terminology of fiber bundles, it was difficult to explain these concepts in a satisfactory way. the situation was further complicated by the fact that Cartan called tangent space which is now known as fiber bundle while the base

space X, being a differentiable manifold, has a tangent space from its differentiable structure,But he saw clearly that the geometrical situation demands the introduction of fiber bundles with rather general fibers." The above reference [5] of Cartan showed that Cartan knew the theory of connection, which is called non-abelian gauge theory, back in 1926.

[30] Shiing-Shen Chern, Harold I. Levine, and Louis Nirenberg, *Intrinsic norms on a complex manifold*, Global Analysis (Papers in Honor of K. Kodaira), 1969, pp. 119–139.

[31] Shiing-Shen Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Mathematica **133** (1974), 219–271.

[32] Shiing-Shen Chern and James Simons, *Characteristic forms and geometric invariants*, Annals of Mathematics. Second Series **99** (1974), 48–69.

[33] Shiing-Shen Chern and E. Spanier, *The homology structure of sphere bundles*, Proceedings of the National Academy of Sciences of the United States of America **36** (1950), 248–255. In this paper, Chern and Spanier proved the Thom isomorphism which is fundamental in differential topology of fiber bundles. Thom's paper appeared in C.R. Paris in the same year.

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[36] Max Eger, *Les systèmes canoniques d'une variété algébrique à plusieurs dimensions*, Annales scientifiques de l'École Normale Supérieure **3e série, 60** (1943), 143–172 (fr).

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[48] André Lichnerowicz, *Théorie globale des connexions et des groupes d'holonomie*, Edizioni Cremonese, Roma, 1957.

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