Rank 2 Weak Fano bundles over Fano 3-folds of Picard rank one

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Preface(?)

- I apologise that there will be no group action in this talk (although Grassmannians appear several times...)
- This talk gives a very quick tour for our classification result, and thus contains too many contents. However each results are in some sense independent and you can start listening from wherever you want :)
- This slide is available in the conference website

Introduction

The main result of today is

The classification of rank two weak Fano bundles
over Fano threefolds of Picard rank one.

Let X be a smooth projective variety (over $\mathbb{C}$).

Definition (Langer)

A vector bundle $\mathcal{E}$ is weak Fano if

$$\mathbb{P}_X(\mathcal{E}) = \mathbf{Proj}(\mathbf{Sym}^{\bullet}\,\mathcal{E})$$

is weak Fano, i.e. $-K_{\mathbb{P}_X(\mathcal{E})}$ is nef and big.

Background

Definition (Szurek-Wisniewski)

A vector bundle $\mathcal{E}$ is Fano if $\mathbb{P}_X(\mathcal{E})$ is Fano.

- Fano 3-folds are successfully classified by Iskovskih and Mori-Mukai.
- The classification of Fano 4-folds becomes much more complicated, and still very far from the completion.
- Rank 2 Fano bundles over Fano 3-folds provides a reasonable class of Fano 4-folds with $b_2 > 1$ to study.
- Rank 2 Fano bundles over Fano 3-folds of Picard rank one are completely classified by Muñoz-Occhetta-Solá Conde.

Muñoz-Occhetta-Solá Conde showed that:

- Up to twist, any rank two Fano bundle over a Fano manifold X with
$$H^2(X;\mathbb{Z}) = H^4(X;\mathbb{Z}) = \mathbb{Z}$$
is the pull-back of rank 2 univ. quot. bdl. under a finite map
$$\psi\colon X \to \mathrm{Gr}(N,2).$$
- They gave the classification of all possibilities for (X,ψ).

Remark

By Szurek-Wiśniewski, Yasutake, and Fujino-Gongyo, it is known that if X admits Fano (resp. weak Fano) bundle, then X is Fano (resp. weak Fano).
In particular, if X with $\rho = 1$ admits a weak Fano bundle, then X is Fano.

Contexts for the weak Fano setting are:

- Takeuchi's 2-ray game method revealed that:
  Weak Fano manifolds with $\rho = 2$ plays an important role in the study of Fano manifolds of $\rho = 1$.
- As a generalisation of the classification of Fano bundles, it is natural to consider the classification of weak Fano bundles.

The aim for today:

To share the classification result for rank 2 weak Fano bundles over Fano threefolds with $\rho = 1$.

Today's talk depends on **3** papers:

[FHI1] T. Fukuoka, W. Hara, D. Ishikawa, *Classification of rank two weak Fano bundles on del Pezzo threefolds of degree four*, Math. Z, 2022.

[FHI2] T. Fukuoka, W. Hara, D. Ishikawa, *Rank two weak Fano bundles on del Pezzo threefolds of degree five*, Intarnat. J. Math, 2023.

[FHI3] T. Fukuoka, W. Hara, D. Ishikawa, *Rank two weak Fano bundles on Fano threefolds of Picard rank one*, preprint, 2025, arXiv:2505.03263

Very rough classification

Let X be a Fano 3-fold with $\rho = 1$.

Theorem A (Fukuoka-H-Ishikawa)

Let $\mathcal{E}$ be a rank two weak Fano bundle over X.

(1) If $c_1(\mathcal{E}) \not\equiv c_1(X) \pmod 2$, then $\mathcal{E}$ is a Fano bundle.

(2) If $c_1(\mathcal{E}) \equiv c_1(X) \pmod 2$, then

$$\mathcal{E}\left(\frac{c_1(X) - c_1(\mathcal{E})}{2}\right)$$

is globally generated, with the only exception when

X is a del Pezzo threefold of degree 1 and $\mathcal{E} \simeq \mathcal{O}_X(c_1(\mathcal{E})/2)^{\oplus 2}$.

- (1) is the consequence of the whole classification that will be described later, and there is no concise proof yet.
- (2) can be shown using techniques from MMP (see [FHI1])

On the projective space $\mathbb{P}^3$

Theorem (Yasutake, Szurek-Wisniewski)

A rank 2 weak Fano bundle $\mathcal{E}$ over $\mathbb{P}^3$ is one of the following (up to twist).

(1) $\mathcal{O} \oplus \mathcal{O}$

(2) $\mathcal{O} \oplus \mathcal{O}(-1)$

(3) $\mathcal{O}(-1) \oplus \mathcal{O}(1)$

(4) $\mathcal{O}(-2) \oplus \mathcal{O}(1)$

(5) $\mathcal{O}(-2) \oplus \mathcal{O}(2)$

(6) The null-correlation bundle $\mathcal{N} = \Omega_{\mathbb{P}^3}(1)/\mathcal{O}(-1)$.

(7) A stable bundle with $(c_1, c_2) = (0, 2)$

(8) A stable bundle with $(c_1, c_2) = (0, 3)$

In addition, all cases have examples.

- Let $M_{0,3}$ be the moduli space of rank 2 stable bundle over $\mathbb{P}^3$ with $(c_1, c_2) = (0, 3)$. It is known by Ellignsrud that $M_{0,3}$ has two connected components $M^0_{0,3}, M^1_{0,3}$ where

$$M^{\alpha}_{0,3} = \{\mathcal{F} \in M_{0,3} \mid h^1(\mathcal{F}(-2)) \equiv \alpha \pmod 2\}.$$

- Yasutake constructed a wF bdl $\in M^0_{0,3}$, and asked:

Question

Does $M^1_{0,3}$ contain a weak Fano bundle?

- Our Theorem A (2) answers negatively to this question, and hence makes Yasutake's classification more precise.
- Indeed, since $\mathcal{E}(2)$ is globally generated by Theorem A (2), a general section $s \in H^0(\mathcal{E}(2))$ has the vanishing locus $C = V(s)$ that is a smooth elliptic curve of degree 7, with ex. seq. $0 \to \mathcal{O}(-4) \to \mathcal{E}(-2) \to \mathcal{I}_{C/X} \to 0$. This yields $H^1(\mathcal{E}(-2)) = 0$ and hence $\mathcal{E} \in M^0_{0,3}$.

On del Pezzo 3-folds

- A Fano 3-fold X is called del Pezzo if $-K_X \sim 2H$.
- $\operatorname{Pic} X = \mathbb{Z}[H]$ iff $1 \leq \deg X = H^3 \leq 5$.
- If $\deg X = 3$, then X is a cubic 3-fold.
- If $\deg X = 4$,
  then X is a smooth intersection of two quadrics in $\mathbb{P}^5$.
- If $\deg X = 5$,
  then X is a codim 3 lin. sect. of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$.

Proposition (Fukuoka-H-Ishikawa [FHI2])

Let X be a dP 3-fold of $\deg X \in \{1, 2\}$. Then all rank 2 weak Fano bundles over X split. Up to twist, they are isom. to one of

(1) $\mathcal{O}^{\oplus 2}$,

(2) $\mathcal{O} \oplus \mathcal{O}(-1)$ or

(3) $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Theorem (Ishikawa)

Let X be a cubic 3-fold, and $\mathcal{E}$ a rk 2 bdl.
Then $\mathcal{E}$ is weak Fano iff (up to twist) it is one of

(1) $\mathcal{O}^{\oplus 2}$

(2) $\mathcal{O} \oplus \mathcal{O}(-1)$

(3) $\mathcal{O}(1) \oplus \mathcal{O}(-1)$

(4) The unique non-triv. ext. $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_{L/X} \to 0$, where $L \subset X$ is a line.

(5) A minimal instanton bundle.

- A bundle in (4) is indecomposable and strictly semistable. Such a case does not happen if X is not del Pezzo.
- Instanton bundles are rank 2 stable bundles $\mathcal{E}$ with $c_1 = 0$ and $H^1(\mathcal{E}(-1)) = 0$, and minimal instanton bundles are those with $c_2 = 2$.

Theorem B (Fukuoka-H-Ishikawa [FHI1])

Let X be a dP 3-fold of $\deg = 4$, and $\mathcal{E}$ a rk 2 bdl.
Then $\mathcal{E}$ is weak Fano iff (up to twist) it is one of

(1) $\mathcal{O}^{\oplus 2}$

(2) $\mathcal{O} \oplus \mathcal{O}(-1)$

(3) $\mathcal{O}(1) \oplus \mathcal{O}(-1)$

(4) The unique non-triv. ext. $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_{L/X} \to 0$, where $L \subset X$ is a line.

(5) The pull-back of the rank two univ. subbdl. by $X \subset \mathbb{Q}^4 \simeq \mathrm{Gr}(2,4)$.

(6) The pull-back of the spinor bundle by $X \xrightarrow{2:1} \mathbb{Q}^3$.

(7) A minimal instanton bundle.

(8) The unique non-triv.ext. $0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{I}_{C/X}(1) \to 0$, where C is an ell. curve of $\deg = 7$ def. by quadratic equations.

In addition, all cases have example.

The most difficult part of this classification is

Proposition (Fukuoka-H-Ishikawa [FHI1])

Any dP 3-fold X of $\deg = 4$ contains an elliptic curve C of $\deg = 7$ def. by quadratic equations. In particular, every X admits rank 2 weak Fano bundle in (8).

Recall that $X \subset \mathbb{P}^5$.

Proposition (Fukuoka-H-Ishikawa [FHI1])

Let $C \subset \mathbb{P}^5$ be an elliptic curve of degree 7. TFAE

(1) C is defined by quadratic equations

(2) C has no trisecant.

This proposition can be shown using Mukai's technique: Construct some vector bundle over $C \times \mathrm{Bl}_C \mathbb{P}^5$, and relate the vanishing of cohomology with the base point freeness of $|2H - E|$ on $\mathrm{Bl}_C \mathbb{P}^5$.

To construct quadratically generated elliptic curve of deg 7,

- Fix a conic $\Gamma \subset X$ and consider hyperplane section $H \subset X$.
- Using a geometry of dP surface H, find a quintic elliptic curve $D \subset H$ that meets Γ transversally at a single point.
- Show $D \cup \Gamma \subset X$ is a nodal elliptic curve without trisecants.
- Show that the local smoothing of the node of $D \cap \Gamma$ extends to the global smoothing C of $D \cap \Gamma$ in X.
- C has no trisecant, either.
- Hence C is quadratically generated by Proposition.

Remark

The wF bundles in (8) has $(c_1, c_2) = (0, 3)$, and stable.
Let $M^{\mathrm{wF}}_{(0,3)}(X)$ be the moduli space of rank 2 weak Fano bundles over X with $(c_1, c_2) = (0, 3)$.
Our result shows $M^{\mathrm{wF}}_{(0,3)}(X) \neq \emptyset$ for all dP 3-fold X of deg 4, but we still don't know if $M^{\mathrm{wF}}_{(0,3)}(X)$ is connected or not.

Theorem C (Fukuoka-H-Ishikawa [FHI2])

Let X be the dP 3-fold of $\deg = 5$, and $\mathcal{E}$ a rk 2 bdl.
Then $\mathcal{E}$ is weak Fano iff (up to twist) it is one of

(1) $\mathcal{O}^{\oplus 2}$

(2) $\mathcal{O} \oplus \mathcal{O}(-1)$

(3) $\mathcal{O}(1) \oplus \mathcal{O}(-1)$

(4) The unique non-triv. ext. $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_{L/X} \to 0$, where $L \subset X$ is a line.

(5) The pull-back of the rank two univ. subbdl. by $X \subset \mathrm{Gr}(2,5)$.

(6) A minimal instanton bundle.

(7) The unique non-triv.ext. $0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{I}_{C/X}(1) \to 0$, where C is an ell. curve of $\deg = 8$ def. by quadratic equations.

(8) The unique non-triv.ext. $0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{I}_{C/X}(1) \to 0$, where C is an ell. curve of $\deg = 9$ def. by quadratic equations.

In addition, all cases have examples.

Remark

This case, X is unique up to isomorphism, and the existence of elliptic curves as in (7) and (8) is known from the lattice theory of a K3 surface $S \in |-K_X|$.

When $\deg X = 5$, the most difficult part is:

Proposition (Fukuoka-H-Ishikawa [FHI2])

If $\mathcal{E}$ is a rk 2 wF bdl. with $c_1 = -1$. Then $\mathcal{E}$ is

(2) $\mathcal{O} \oplus \mathcal{O}(-1)$ $(c_2 = 0)$, or

(5) the restriction of the rank two univ. subbdl. by $X \subset \mathrm{Gr}(2,5)$ $(c_2 = 2)$.

- By Riemann-Roch $\chi(\mathcal{E}) = 1 - \frac{1}{2}c_2$, and hence c_2 is even.
- Since $-K_{\mathbb{P}(\mathcal{E})}$ is nef and big, $(-K_{\mathbb{P}(\mathcal{E})})^4 > 0$ and this implies $c_2 \leq 4$.
- Thus we must exclude the case $(c_1, c_2) = (-1, 4)$.

- If $\mathcal{E}$ is a weak Fano bundle with $c_1 = -1$, then $\mathcal{E}(2)$ is ample and $c_1(\mathcal{E}(2)) = 3$.
- Thus for any line $l \subset X$, $\mathcal{E}(2)|_l \simeq \mathcal{O}_l(1) \oplus \mathcal{O}_l(2)$.

(∗) Therefore $\mathcal{E}|_l \simeq \mathcal{O}_l(-1) \oplus \mathcal{O}_l$.

- Let $\mathbf{Hilb}_{t+1}(X) \simeq \mathbb{P}^2$ be the Hilbert scheme of lines, and U the univ. fam. with the projections $X \xleftarrow{e} U \xrightarrow{\pi} \mathbb{P}^2$.
- $H \in |e^*\mathcal{O}_X(1)|$, $L \in |\pi^*\mathcal{O}_{\mathbb{P}^2}(1)|$.
- $\pi_*e^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(-a)$ by Grauert and $(*)$.
- $0 \to \pi^*\mathcal{O}(-a) \to e^*\mathcal{E} \to e^*\mathcal{E}/\pi^*\mathcal{O}(-a) \to 0$, and $e^*\mathcal{E}/\pi^*\mathcal{O}(-a) \simeq \mathcal{O}(-H + aL)$.
- This gives $e^*c_2(\mathcal{E}) = c_2(e^*\mathcal{E}) = aHL - a^2L$.
- $e^*c_2(\mathcal{E})$ should be divisible by $e^*l \sim HL - 2L^2$.
- Thus $a = 0, 2$ and hence $c_2(\mathcal{E}) = 0, 2$.
- If $c_2(\mathcal{E}) = 0$, then $\mathcal{E}$ is (2), and if $c_2(\mathcal{E}) = 2$, then $\mathcal{E}$ is (5).

In contract to the case $\deg X \leq 4$,
the dP 3-fold X of $\deg 5$ admits full strong exceptional collection

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) = \langle \mathcal{O}(-1), \mathcal{Q}(-1), \mathcal{R}, \mathcal{O} \rangle$$

by Orlov, where $\mathcal{R}$ and $\mathcal{Q}$ are the restriction of the univ. subbdl. and the univ. quot. bdl. under $X \subset \mathrm{Gr}(2,5)$.

- Put $\mathcal{T} := \mathcal{O}(-1) \oplus \mathcal{Q}(-1) \oplus \mathcal{R} \oplus \mathcal{O}$, then

  $$\Phi := \mathrm{RHom}(\mathcal{T}, -) \colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \mathrm{End}(\mathcal{T}))$$

  is an equivalence.
- If a bundle $\mathcal{F}$ satisfies $\Phi(\mathcal{F}) \in \mathrm{mod}\, \mathrm{End}(\mathcal{T})$, one can consider a projective resolution of $\Phi(\mathcal{F})$.
- Indecomp. projective right $\mathrm{End}(\mathcal{T})$-mods. are given by the image of

  $$\mathcal{O}(-1), \mathcal{Q}(-1), \mathcal{R}, \mathcal{O}$$

Proposition (Fukuoka-H-Ishikawa [FHI2])

Rank 2 wF bdls in (4),(6),(7),(8) fit in ex. seqs.

(4) $0 \to \mathcal{Q}(-1) \to \mathcal{O} \oplus \mathcal{R}^{\oplus 2} \to \mathcal{E} \to 0.$

(6) $0 \to \mathcal{Q}(-1)^{\oplus 2} \to \mathcal{R}^{\oplus 4} \to \mathcal{E} \to 0.$

(7) $0 \to \mathcal{O}(-1) \oplus \mathcal{Q}(-1) \to \mathcal{R}^{\oplus 5} \to \mathcal{O}^{\oplus 8} \to \mathcal{E}(1) \to 0.$

(8) $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{Q}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 6} \to \mathcal{E}(1) \to 0.$

- Computation of the resolution is difficult when in (7) or (8).
- For example, for (8) we had to show $\mathbf{Hom}(\mathcal{Q}(-1), \mathcal{E}) = 0$, which was actually the most difficult part.
- Global generation of $\mathcal{E}(1)$ from Theorem A (2) shows that $\mathcal{E}$ in (6,7,8) is an instanton bundle.
- Moduli space for (6) and (7) are known by Kuznetsov and Sanna.
- The proposition above can be applied to study the moduli for (8) as follows.

- The bundle in (8) has $(c_1, c_2) = (0, 4)$.
- Let $M^{\mathrm{wF}}_{(0,4)}(X)$ be the moduli of those wF bdls.
- Let $\mathcal{K} := \mathrm{Ker}(\mathcal{O}^{\oplus 6} \to \mathcal{E}(1))$.
- Then $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{Q}(-1)^{\oplus 2} \to \mathcal{K} \to 0$ by Prop.
- Let $\mathcal{A} := \langle \mathcal{O}(-1), \mathcal{Q}(-1) \rangle \subset \mathrm{D^b}(\mathrm{coh}\, X)$. Then $\mathcal{K} \in \mathcal{A}$.
- By tilting theory, $\mathcal{A} \simeq \mathrm{D^b}(\mathrm{mod}\, \mathbb{C}\, Q)$, where Q is the 5-Kronecker quiver.
- In addition, $\mathcal{O}(-1)[1]$ and $\mathcal{Q}(-1)$ give all simple modules.
- Since $\mathcal{Q}(-1)^{\oplus 2} \to \mathcal{K} \to \mathcal{O}(-1)^{\oplus 2}[1]$ in $\mathcal{A}$,

$$\underline{\mathrm{dim}} \mathcal{K} = (2, 2).$$

- One can show that $\mathcal{K}$ is stable as a Q-representation.
- Let $M^{\mathrm{st}}_{(2,2)}(Q)$ be the moduli of Q-reps. with $\underline{\mathrm{dim}} = (2, 2)$.
- The correspondence $\mathcal{E} \mapsto \mathcal{K}$ gives an open immersion

$$M^{\mathrm{wF}}_{(0,4)}(X) \hookrightarrow M^{\mathrm{st}}_{(2,2)}(Q).$$

On the quadric

Theorem D (Fukuoka-H-Ishikawa [FHI3])

Let $X = \mathbb{Q}^3$ be the quadric 3-fold, and $\mathcal{E}$ a rk 2 bdl.
Then $\mathcal{E}$ is weak Fano iff (up to twist) it is one of

(1) $\mathcal{O}^{\oplus 2}$

(2) $\mathcal{O} \oplus \mathcal{O}(-1)$

(3) $\mathcal{O}(1) \oplus \mathcal{O}(-1)$

(4) $\mathcal{O}(1) \oplus \mathcal{O}(-2)$

(5) The pull-back of the null-correlation bdl. by lin. proj. $X \xrightarrow{2:1} \mathbb{P}^3$.

(6) The spinor bundle $\mathcal{S}$.

(7) The restriction of a Cayley bundle by $X \subset \mathbb{Q}^5$.

(8) $0 \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}(-1)^{\oplus 10} \to \mathcal{S}^{\oplus 5} \to \mathcal{E}(1) \to 0.$

(9) $0 \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}(-1)^{\oplus 7} \to \mathcal{O}^{\oplus 7} \to \mathcal{E}(2) \to 0.$

In addition, all cases have example.

The Chern classes (c_1, c_2) of each examples are:

(5) $(c_1, c_2) = (0, 2)$ (The pull-back of the nullcorrelation bdl.)

(6) $(c_1, c_2) = (-1, 1)$ (The spinor)

(7) $(c_1, c_2) = (-1, 2)$ (The restriction of a Cayley bundle)

(8) $(c_1, c_2) = (-1, 3)$

(9) $(c_1, c_2) = (-1, 4)$

- If $c_1 = 0$, the Riemann-Roch formula is $\chi(\mathcal{E}) = -\frac{3}{2}c_2 + 2$, and hence c_2 should be even.
- The nef big property of $-K_{\mathbb{P}(\mathcal{E})}$ shows $(-K_{\mathbb{P}(\mathcal{E})})^4 = 48(-2c_2 + 9) > 0$, and hence $c_2 \leq 4$.

Thus Theorem D implicitly contains:

Proposition (Fukuoka-H-Ishikawa [FHI3])

A rank two bdl. $\mathcal{E}$ on $\mathbb{Q}^3$ is NOT weak Fano if $(c_1, c_2) = (0, 4)$

Outline of the proof of the proposition:

- Let $\mathcal{E}$ be a rank 2 bdl. on $\mathbb{Q}^3$ with $(c_1, c_2) = (0, 4)$.
- Put $Y := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{Q}^3$, and let ξ be the taut'l. div.
- By Sols-Szurek-Wiśniewski, $\exists \Gamma_0 \subset Y$ such that
  - (a) $(-K_Y).\Gamma_0 \leq 0$ and
  - (b) $\Gamma = \pi(\Gamma_0) \subset \mathbb{Q}^3$ is a conic
- Assume for contradiction that $Y := \mathbb{P}(\mathcal{E})$ is weak Fano
- Then $(-K_Y).\Gamma_0 = 0$ (since $-K_Y$ is nef).
- The contraction $Y \to \overline{Y}$ associated to $-K_Y$ has at most 1-dim'l fibers (by numerical computations)
- Thus Γ_0 is a smooth rational curve
- Fix an ample $A := \frac{1}{2}(-K_Y + H)$ for $H \in |\pi^* \mathcal{O}_{\mathbb{Q}^3}(1)|$.
- Then $A.\Gamma_0 = \frac{1}{2} H.\Gamma_0 = \frac{1}{2} \mathcal{O}_{\mathbb{Q}^3}(1) \pi(\Gamma_0) = 1$.
- Since Γ_0 is smooth rat. curve, $\dim_{[\Gamma_0]} \mathrm{Hilb}(Y) \geq 1$.

- Thus $\exists$ an smooth proj. curve C with $o \in C$ such that
- there is a diagram

$$\begin{array}{ccccc} S & \xrightarrow{f} & Y & \xrightarrow{\pi} & \mathbb{Q}^3 \\ \downarrow{\scriptstyle p} & & & & \\ C & & & & \end{array}$$

- where f is gen. finite and $f\colon p^{-1}(o) \xrightarrow{\simeq} \Gamma_0$ ("bend" of Γ_0)
- Since $A.\Gamma_0 = 1$, "break" cannot happen.
- In other words, p is a $\mathbb{P}^1$-fibration
- $p\colon S \to C$ is a smooth conic bundle of $\mathbb{Q}^3$.
- Thus $\exists g\colon C \to \mathrm{Hilb}_{2t+1}(\mathbb{Q}^3) \simeq \mathrm{Gr}(3,5)$ finite morphism
- Let $\Delta := \{[\gamma] \in \mathrm{Hilb}_{2t+1}(\mathbb{Q}^3) \mid \gamma \text{ singular}\} \subset \mathrm{Gr}(3,5)$.
- $g(C) \cap \Delta = \emptyset$ (since $p\colon S \to C$ is a smooth conic bdl.)
- This is a contradiction since Δ is an ample divisor.

On Mukai 3-folds

Let X be a Fano 3-fold with $\mathrm{Pic}\, X \simeq \mathbb{Z}[-K_X]$.

Theorem E (Fukuoka-H-Ishikawa [FHI3])

Put $g := \frac{1}{2}((-K_X)^3 + 2)$, and let $\mathcal{E}$ be a rk 2 bdl.
Then $\mathcal{E}$ is weak Fano iff (up to twist) it is one of

(1) $\mathcal{O}^{\oplus 2}$

(2) $\mathcal{O} \oplus \mathcal{O}(-K_X)$

(3) A globally generated vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c_1(X)$ and

$$\lfloor \frac{g+3}{2} \rfloor \leq -K_X.c_2(\mathcal{F}) \leq g - 2.$$

In addition, all cases have example.

The case (3) happens only when $6 \leq g(\leq 12, g \neq 11)$.

Theorem E has two contents:

(A) If $c_1(\mathcal{E})$ is even, then $\mathcal{E} \simeq \mathcal{O}(c_1/2)^{\oplus 2}$.

(B) Construction of examples for all cases in (3).

- For (A), we use "bend-and-break"+"conic bundle" method
- For (B), we use the recent result by Ciliberto-Flamini-Knutsen.
- CFK studied elliptic (normal) curves on X in the context of ACM bundles
- Choose a general elliptic normal curve $C \subset X$ with

$$\lfloor \frac{g+3}{2} \rfloor \leq -K_X.C \leq g-2,$$

and consider the bundle $\mathcal{F}$ that fits in

$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{I}_{C/X}(-K_X) \to 0$$

- Then we can show that $\mathcal{F}$ is nef (and hence weak Fano)
- The method is very numerical, with some projective geometry observations for C.
- A key gadget is the Brill-Noether property of K3 surfaces associated to du Val members of $|-K_X|$.

Theorem A implies the following.

Corollary (Fukuoka-H-Ishikawa [FHI3])

X Fano 3-fold with $\mathrm{Pic}\, X \simeq \mathbb{Z}[-K_X]$, $g = \frac{1}{2}((-K_X)^3 + 2)$.
For any $\lfloor \frac{g+3}{2} \rfloor \leq d \leq g-2$, $\exists$elliptic curve $C \subset X$ such that

(1) $(-K_X).C = d$ and

(2) $\mathcal{I}_{C/X}(-K_X)$ is globally generated.

Note that the inequality makes sense only when $g \geq 6$, and in this case $-K_X$ is very ample (X is a prime Fano 3-fold).

The classification of rank $\mathbf{2}$ weak Fano bundles over Fano $\mathbf{3}$-folds of $\rho = 1$ is now over!!

(a) $\mathbb{P}^{\mathbf{3}}$ (by Yasutake, [FHI1])

(b) $\mathbb{Q}^{\mathbf{3}}$ (by [FHI3])

(c) del Pezzo $\mathbf{3}$-folds (5 families, by Ishikawa and [FHI1, FHI2])

(d) Mukai threefolds (10 families, by [FHI3])

Next To Do

Study the geometry of all weak Fano $\mathbf{4}$-folds $\mathbb{P}(\mathcal{E})$ (2-ray game)

Embedding theorem

- Let X be a Fano 3-fold with $\operatorname{Pic} X \simeq \mathbb{Z}[-K_X]$,
- and $\mathcal{F}$ a rank two weak Fano bundle with $c_1(\mathcal{F}) = c_1(X)$.
- Since $\mathcal{F}$ is globally generated by Theorem A,

$$\exists \Psi \colon X \to \mathrm{Gr}(H^0(\mathcal{F}), 2)$$

such that $\mathcal{F} \simeq \Psi^* \mathcal{Q}_{\mathrm{Gr}(H^0(\mathcal{F}),2)}$.

Theorem F (Fukuoka-H-Ishikawa [FHI3])

Ψ is a closed immersion except when $(X, \mathcal{F})$ is one of

(1) $f \colon X \xrightarrow{2:1} \mathbb{P}^3$ and $\mathcal{F} \simeq f^*(\mathcal{O} \oplus \mathcal{O}(1))$.

(2) $f \colon X \xrightarrow{2:1} \mathbb{Q}^3$ and $\mathcal{F} \simeq f^*(\mathcal{O} \oplus \mathcal{O}(1))$.

(3) $f \colon X \xrightarrow{2:1} V_5 \subset \mathrm{Gr}(5,2)$ and $\mathcal{F} \simeq f^*(\mathcal{Q}_{\mathrm{Gr}(5,2)}|_{V_5})$.

Thank you very much!