

Euler System of CM Points on Shimura Curves

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Chapter 1

Introduction

1.1 The Main Results

Let A be an abelian variety defined over a number field F , K a finite extension of F . The Birch and Swinnerton-Dyer (BSD, in short) conjecture for (A, K) predicts that the rank of the Mordell-Weil group $A(K)$ is equal to the vanishing order at $s = 1$ of the L-function $L(s, A/K)$ of A over K :

$$\text{rank}_{\mathbb{Z}} A(K) = \text{ord}_{s=1} L(s, A/K). \quad (1)$$

There is a refined version of the BSD conjecture on the leading coefficient of the Taylor expansion at $s = 1$ of the L-function in term of arithmetic invariants of A (see §1.2). In this book, we give some evidence to these conjectures.

Assume that A is an abelian variety over a totally real field F , and is associated with an automorphic form ϕ for $\text{GL}_{2,F}$ of weight $(2, \dots, 2)$, conductor N , and with trivial central character, in the sense that

$$L_v(s, A) = \prod_{\sigma} L_v(s - \frac{1}{2}, \phi^{\sigma}),$$

for all finite places v of F , where ϕ^{σ} are all distinct conjugations of ϕ under $\sigma \in \text{Aut}(\mathbb{C})$. Then A has real multiplication over F by the subring $\mathbb{Z}[\phi]$ of \mathbb{C} generated over \mathbb{Z} by Hecke eigenvalues of ϕ . By an isogenous, we may assume A has full real multiplication by \mathcal{O}_{ϕ} , the integral closure of $\mathbb{Z}[\phi]$. Let K/F be a totally imaginary quadratic extension with discriminant prime to N . Using the generalization of Gross-Zagier formula to totally real fields by the second author [24], we know that if $\text{ord}_{s=1/2} L(s, \phi_K) \leq 1$ then

$$\text{ord}_{s=1} L(s, A/K) = [\mathcal{O}_{\phi} : \mathbb{Z}] \cdot \text{ord}_{s=1/2} L(s, \phi_K). \quad (2)$$

Theorem 1.1.1. *If $\text{ord}_{s=1/2} L(s, \phi_K) = 1$, then $\text{rank}_{\mathbb{Z}} A(K) = \text{ord}_{s=1} L(s, A/K)$ and the Shafarevich-Tate group $\text{III}(K, A)$ is finite.*

If $\text{ord}_{s=1/2} L(s, \phi_K) = 0$ and A does not have complex multiplication, then $A(K)$ is finite. If furthermore, A is geometrically simple, then $\text{III}(K, A)$ is also finite.

With a non-vanishing result [10], we have