Euler System of CM Points on Shimura Curves

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Chapter 1

Introduction

1.1 The Main Results

Let A be an abelian variety defined over a number field F, K a finite extension of F. The Birch and Swinnerton-Dyer (BSD , in short) conjecture for (A, K) predicts that the rank of the Mordell-Weil group $A(K)$ is equal to the vanishing order at $s = 1$ of the L-function $L(s, A/K)$ of A over K:

$$\mathrm{rank}_{\mathbb{Z}} A(K) = \mathrm{ord}_{s=1} L(s, A/K). \tag{1}$$

There is a refined version of the BSD conjecture on the leading coefficient of the Taylor expansion at $s = 1$ of the L-function in term of arithmetic invariants of A (see §1.2). In this book, we give some evidence to these conjectures.

Assume that A is an abelian variety over a totally real field F, and is associated with an automorphic form ϕ for $\mathrm{GL}_{2,F}$ of weight $(2, \cdots, 2)$, conductor N, and with trivial central character, in the sense that

$$L_v(s, A) = \prod_{\sigma} L_v(s - \frac{1}{2}, \phi^\sigma),$$

for all finite places v of F, where ϕ^σ are all distinct conjugations of ϕ under $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then A has real multiplication over F by the subring $\mathbb{Z}[\phi]$ of $\mathbb{C}$ generated over $\mathbb{Z}$ by Hecke eigenvalues of ϕ. By an isogenous, we may assume A has full real multiplication by $\mathcal{O}_\phi$, the integral closure of $\mathbb{Z}[\phi]$. Let K/F be a totally imaginary quadratic extension with discriminant prime to N. Using the generalization of Gross-Zagier formula to totally real fields by the second author [24], we know that if $\mathrm{ord}_{s=1/2} L(s, \phi_K) \leq 1$ then

$$\mathrm{ord}_{s=1} L(s, A/K) = [\mathcal{O}_\phi : \mathbb{Z}] \cdot \mathrm{ord}_{s=1/2} L(s, \phi_K). \tag{2}$$

Theorem 1.1.1. *If* $\mathrm{ord}_{s=1/2} L(s, \phi_K) = 1$, *then* $\mathrm{rank}_{\mathbb{Z}} A(K) = \mathrm{ord}_{s=1} L(s, A/K)$ *and the Shafarevich-Tate group* $Ш(K, A)$ *is finite.*

If $\mathrm{ord}_{s=1/2} L(s, \phi_K) = 0$ *and* A *does not have complex multiplication, then* $A(K)$ *is finite. If furthermore,* A *is geometrically simple, then* $Ш(K, A)$ *is also finite.*

With a non-vanishing result [10], we have