p-ADIC COHOMOLOGY AND CLASSICALITY OF OVERCONVERGENT HILBERT MODULAR FORMS

by

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Abstract. — Let F be a totally real field in which p is unramified. We prove that, if a cuspidal overconvergent Hilbert cuspidal form has small slopes under U_p-operators, then it is classical. Our method follows the original cohomological approach of Coleman. The key ingredient of the proof is giving an explicit description of the Goren-Oort stratification of the special fiber of the Hilbert modular variety. A byproduct of the proof is to show that, at least when p is inert, of the rigid cohomology of the ordinary locus has the same image as the classical forms in the Grothendieck group of Hecke modules.

Contents

1. Introduction

The classicality results for p-adic overconvergent modular forms started with the pioneer work of Coleman [**Co96**], where he proved that an overconvergent modular form of weight k and slope $< k-1$ is actually classical. Coleman proved his theorem using p-adic cohomology and an ingenious dimension counting argument. Later on, Kassaei [**Ks06**] reproved Coleman's theorem based on an analytic continuation result by Buzzard [**Bu03**]. In the Hilbert case, Sasaki [**Sas10**] proved classicality of small slope overconvergent Hilbert modular forms when the prime p is totally split in the concerning totally real field. With a less optimal slope condition, such classicality result for overconvergent Hilbert modular forms was proved by the first named author [**Ti11**] in the quadratic inert case, and by Pilloni and Stroh in the general unramified case [**PS11**]. The methods of [**Sas10, Ti11, PS11**] followed that of Kassaei, and used the analytic continuation of overconvergent Hilbert modular forms.

In this paper, we will follow Coleman's original cohomological approach to prove the classicality of cuspidal overconvergent Hilbert modular forms. Let us describe in detail our main results. We fix a prime number $p \geq 2$. Let F be a totally real field of degree $g = [F : \mathbb{Q}] \geq 2$ in which p is unramified, and denote by $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the primes of F above p.

2000 ***Mathematics Subject Classification.*** — 11F41, 11F33, 14F30.
Key words and phrases. — p-adic modular forms, Hilbert modular varieties, rigid cohomology.

Let Σ_∞ be the set of archimedean places of F. We fix an isomorphism $\iota_p : \mathbb{C} \cong \overline{\mathbb{Q}}_p$. For each $\mathfrak{p}_i$, we denote by $\Sigma_{\infty/\mathfrak{p}_i}$ the archimedean places $\tau \in \Sigma_\infty$ such that $\iota_p \circ \tau$ induce the prime $\mathfrak{p}_i$. We fix an ideal $\mathfrak{N}$ of $\mathcal{O}_F$ coprime to p We consider the following level structures:

$$K_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) | a \equiv 1, c \equiv 0 \pmod{\mathfrak{N}} \right\};$$

$$K_1(\mathfrak{N})^p \mathrm{Iw}_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1(\mathfrak{N}) | c \equiv 0 \pmod{p} \right\}.$$

Consider a multiweight $(\underline{k}, w) \in \mathbb{Z}^{\Sigma_\infty} \times \mathbb{Z}$(1) such that $w \geq k_\tau \geq 2$ and $k_\tau \equiv w \pmod 2$ for all τ. Our first main theorem is the following

***Theorem 1* (Theorem 6.9).** — *Let f be a cuspidal overconvergent Hilbert modular form of multiweight $(\underline{k}, w)$ and level $K_1(\mathfrak{N})$, which is an eigenvector for all Hecke operators. Let $\lambda_{\mathfrak{p}_i}$ denote the eigenvalue of f for the operator $U_{\mathfrak{p}_i}$ for $1 \leq i \leq r$. If the p-adic valuation of each $\lambda_{\mathfrak{p}_i}$ satisfies*

$$\mathrm{val}_p(\lambda_{\mathfrak{p}_i}) < \sum_{\tau \in \Sigma_{\infty/\mathfrak{p}_i}} \frac{w - k_\tau}{2} + \min_{\tau \in \Sigma_{\infty/\mathfrak{p}_i}} \{k_\tau - 1\}, \tag{1.0.1}$$

then f is a classical (cuspidal) Hilbert modular eigenform of level $K_1(\mathfrak{N})^p \mathrm{Iw}_p$.

Here the p-adic valuation val_p is normalized so that $\mathrm{val}_p(p) = 1$. The term $\sum_\tau \frac{w-k_\tau}{2}$ is a normalizing factor that appears in the definition of cuspidal overconvergent Hilbert modular forms; any cuspidal overconvergent Hilbert eigenform has $U_{\mathfrak{p}_i}$-slope greater than or equal to this quantity. Up to this normalizing factor, Theorem 1 was proved in [**PS11**] (and also in [**Ti11**] for the quadratic case) with slope bound $\mathrm{val}_p(\lambda_{\mathfrak{p}_i}) < \sum_{\tau \in \Sigma_{\mathfrak{p}_i}} \frac{w-k_\tau}{2} + \min_{\tau \in \Sigma_{\infty/\mathfrak{p}_i}} (k_\tau - [F_{\mathfrak{p}_i} : \mathbb{Q}_p])$. The slope bound (1.0.1), believed to be optimal, was conjectured by Breuil in a unpublished note [**Br11-**], which inspires this work a lot. Actually, in Theorem 6.9, we also give some classicality results using theta operators if the slope bound (1.0.1) is not satisfied, as conjectured by Breuil in *loc. cit.* Finally, Christian Johansson [**Jo12**] also obtained independently in his thesis similar results for overconvergent automorphic forms for rank two unitary group, but under a even less optimal slope bound.

We now explain the proof of our theorem. As in [**Co96**], the first step is to relate the cuspidal overconvergent Hilbert modular forms to a certain p-adic cohomology group of the Hilbert modular variety.

We take the level structure K to be hyperspecial at places above p; and $K^p\mathrm{Iw}_p$ the corresponding level structure with Iwahori group at all places above p. Let $\mathbf{X}$ be the integral model of the Hilbert modular variety of level K defined over the ring of integers of a finite extension L over $\mathbb{Q}_p$. We choose a toroidal compactification $\mathbf{X}^{\mathrm{tor}}$ of $\mathbf{X}$. Let X^{tor} and X denote respectively the special fibers of $\mathbf{X}^{\mathrm{tor}}$ and $\mathbf{X}$ over $\overline{\mathbb{F}}_p$, and D be the boundary $X^{\mathrm{tor}} - X$. Let $X^{\mathrm{tor,ord}}$ be the ordinary locus of X^{tor}. Let $\mathscr{F}^{(\underline{k},w)}$ denote the corresponding overconvergent log-F-isocrystal sheaf of multiweight $(\underline{k}, w)$ on X^{tor}, and let $S^\dagger_{(\underline{k},w)}$ denote the space of cuspidal overconvergent Hilbert modular forms. We consider the rigid cohomology of $\mathscr{F}^{(\underline{k},w)}$ over the ordinary locus of X^{tor} with compact support at cusps, denoted by $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ (see Subsection 3.4 for its precise definition). Using the dual BGG-complex and a cohomological computation due to Coleman [**Co96**], we show in Theorem 3.5 that, the cohomology group above is computed by a complex consisting of cuspidal overconvergent Hilbert modular forms.

(1)In this paper, we choose a convention on weight that is better adapted for arithmetic applications. So each archimedean component of the automorphic representation associated to a cuspidal Hilbert eigenform of multiweight $(\underline{k}, w)$ has central character $t \mapsto t^{w-2}$. This agrees with [**Sa09**].

Let us explain more explicitly this result in the case when F is a real quadratic field and p is a prime inert in $F/\mathbb{Q}$. Then the result of Theorem 3.5 says that the cohomology group $H^{\star}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ (together with its Hecke action) is computed by the complex

$$\mathscr{C}^{\bullet}\colon\ S^{\dagger}_{(2-k_1,2-k_2,w)} \xrightarrow{(\Theta_1,\Theta_2)} S^{\dagger}_{(k_1,2-k_2,w)} \oplus S^{\dagger}_{(2-k_1,k_2,w)} \xrightarrow{-\Theta_2\oplus\Theta_1} S^{\dagger}_{(k_1,k_2,w)},$$

where the Θ_i's are essentially (k_i-1)-times composition of the Hilbert analogues of the well-known θ-operator for the elliptic modular forms. We refer the reader to Subsection 2.15 and Remark 2.17 for the precise expression of Θ_i's, and to (3.3.3) for the definition of the complex $\mathscr{C}^{\bullet}$ in the general case. Here we emphasize that the natural construction of the complex $\mathscr{C}^{\bullet}$ is automatically Hecke equivariant; this Hecke action on its terms $S^{\dagger}_{\star}$ differs from those given in [**KL05**] by a twist, except for the last term $\star = (k_1, k_2)$. This can be seen using the explicit formulas for Θ_i, as given in Remark 2.17. An important fact for us is that, under this new Hecke action on $S^{\dagger}_{\star}$, the slope condition (1.0.1) can be satisfied only for eigenforms in the last term $S^{\dagger}_{(k_1,k_2)}$. In other words, if an eigenform $f \in S^{\dagger}_{(k_1,k_2,w)}$ satisfies the slope condition, then it has nontrivial image in the cohomology group $H^{g}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$. This result on U_p-action is explained in Proposition 3.24. Moreover, the cohomological approach allows us to prove the following strengthen version of Theorem 1: if a cuspidal overconvergent Hilbert modular form f of multiweight $(\underline{k}, w)$ and level K does not lie in the image of all Θ-maps, then f is a classical (cuspidal) Hilbert modular form.

The second step of the proof of Theorem 1 is to compute $H^{\star}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ using the Goren-Oort stratification of X. A key ingredient here is the explicit description of these GO-strata of X given in [**TX13a**]. In the quadatic inert case considered above, the main results of [**TX13a**] can be described as follows. Let X_1 and X_2 be respectively the vanishing loci of the two partial Hasse-invariants on X^{tor}. Then the ordinary locus $X^{\mathrm{tor,ord}} \subseteq X^{\mathrm{tor}}$ is the complement of the union $X_1 \cup X_2$ of the two divisors. Put $X_{12} = X_1 \cap X_2$. The subvarieties X_1, X_2, and X_{12} are known to be proper smooth subvarieties of X^{tor} which do not meet the cusps. The main result of [**TX13a**] says that X_1 and X_2 are both $\mathbb{P}^1$-bundles over $\mathbf{Sh}_K(B^{\times}_{\infty_1,\infty_2})_{\overline{\mathbb{F}}_p}$, the special fiber of the discrete Shimura variety of level K associated to the quarternion algebra B_{∞_1,∞_2} over F ramified exactly at both archimedean places. Their intersection X_{12} may be identified with the Shimura variety $\mathbf{Sh}_{K^p\mathrm{Iw}_p}(B^{\times}_{\infty_1,\infty_2})_{\overline{\mathbb{F}}_p}$ for the same group but with Iwahori level structure at p. Moreover, these isomorphisms are compatible with tame Hecke actions.

In the general case, for each subset $\mathtt{T} \subset \Sigma_{\infty}$, one has the closed GO-stratum $X_{\mathtt{T}}$ defined as the vanishing locus of the partial Hasse invariants corresponding to $\mathtt{T}$. This is a proper and smooth closed subvariety of X of codimension $\#\mathtt{T}$. The main result of [**TX13a**] shows that $X_{\mathtt{T}}$ is a certain $(\mathbb{P}^1)^N$-bundle over the special fiber of another quaternionic Shimura variety. In fact, this result is more naturally stated for the Shimura variety associated to the group $\mathrm{GL}_2(F) \times_{F^{\times}} E^{\times}$ with E a quadratic CM extension. We refer the reader to Section 5 for more detailed discussion. Using this result and the Jacquet-Langlands correspondence, one can compute the cohomology of each closed GO-stratum. General formalism of rigid cohomology then produces a spectral sequence which relates the desired cohomology group $H^{\star}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ to those of closed GO-strata. In the general case, we prove the following

Theorem 2 **(Theorems 3.5 and 6.1)**. — *For a multiweight $(\underline{k}, w)$, we have the following equalities of modules in the Grothendieck group of modules of the tame Hecke algebra $\mathscr{H}(K^p, L)$:*

$$\sum_{J\subseteq\Sigma_{\infty}} (-1)^{\#J}\big[(S^{\dagger}_{\epsilon_J(\underline{k},w)})^{\mathrm{slope}\le T}\big] = \big[H^{\star}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})\big] = (-1)^{g}\big[S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)\big],$$

for T sufficiently large, where

- ϵ_J *takes* k_τ *to itself if* $\tau \in J$ *and to* $2-k_\tau$ *if* $\tau \notin J$ *(for the precise definition of* $S^\dagger_{\epsilon_J(\underline{k},w)}$*, see* (3.3.2) *and* (2.15.1)*);*
- *the superscript slope*$\leq T$ *means to take the* finite dimensional *subspace where the slope of the product of the* $U_\mathfrak{p}$*-operators is less than or equal to* T*; and*
- $S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)$ *is the space of classical cuspidal Hilbert modular forms of level* $K^p\mathrm{Iw}_p$*.*

At this point, there are two ways to proceed to get Theorem 1. The first approach is unconditional. We first use Theorem 2 to prove the classicality result when the slope is much smaller the weight (Proposition 6.3). Then we improve the slope bound by studying global crystalline periods over the eigenvarieties (Theorem 6.9). In fact, we can prove something much stronger: if an eigenform f does not lie in the image of the Θ-maps in the complex $\mathscr{C}^\bullet$, then f is a classical Hilbert modular eigenform (Theorem 6.9). This approach, to some extent, relies on the strong multiplicity one of overconvergent Hilbert modular forms. This approach is explained in Section 6.

The second approach is more involved, and we need to assume

(1) either p is inert in F, i.e. p stays as a prime in $\mathcal{O}_F$,
(2) or the action of the "partial Frobenius" on the cohomology of quaternionic Shimura variety are as expected by general Langlands conjecture. (See Conjecture 5.18)

We defer the definition of partial Frobenius to the context of the paper. Under this assumption, we can strengthen Theorem 2 as

Theorem 3. — *Assume the assumption above, the equality in Theorem 2 is an equality in the Grothendieck group of modules of* $\mathscr{H}(K^p, L)[U^2_\mathfrak{p}; \mathfrak{p} \in \Sigma_p]$*.*

This theorem is proved in Section 7, using a combinatorially complicated argument.

The reason that Theorem 3 is stated for the action of squares of $U_\mathfrak{p}$ is the following: the description of GO strata is proved in [**TX13a**] using unitary Shimura varieties, where only the twisted partial Frobenius (instead of partial Frobenius itself) has a group theoretic interpretation, which is, morally, the Hecke operator given by $\left(\begin{smallmatrix} \varpi_\mathfrak{p} & 0 \\ 0 & \varpi_\mathfrak{p}^{-1} \end{smallmatrix}\right)$, where $\varpi_\mathfrak{p}$ denotes the idèle of F which is a uniformizer at $\mathfrak{p}$ and 1 at other places. One might be able to fix this small defect by modify the PEL type unitary moduli problem slightly differently to define the moduli problem for $\mathrm{GL}_{2,F} \times_{F^\times} E^\times$ directly. However, we do not plan to pursue this approach.

Now it is a trivial matter to deduce Theorem 1 from Theorem 3 (under the assumption above). In fact, we only need f to be a generalized eigenvector for all the $U_{\mathfrak{p}_i}$-operators satisfying the slope condition (i.e. f does not have to be an eigenvector for the tame Hecke actions). The upside of this approach is that one may avoid using the strong multiplicity one for overconvergent Hilbert modular forms. This is crucial when studying other quarternionic Shimura varieties when strong multiplicity one is not available. Moreover, Theorem 3 has its own interest; as it gives a quite concrete description of the rigid cohomology of the ordinary locus.

Another related intriguing question is whether $H^*_{\mathrm{rig}}(X^{\mathrm{tor,ord}}_{k_0}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ is concentrated in degree g. We shall address this question in the forthcoming paper [**TX13b**]. It turns out that the result depends on the Satake parameter at p of the corresponding automorphic representation.

Our consideration of the cohomology group $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}_{k_0}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ was stimulated by a conversation with Kaiwen Lan in December 2011, and he explained to us the vanishing result Lemma 3.6. After we finished the first version of the paper and talked to Lan again, we found htat the same cohomology group (in a more general context) was used in his recent joint work with M. Harris, R. Taylor and J. Thorne [**HLTT13**].

Structure of the paper. — Section 2 reviews basic facts about Hilbert modular varieties, as well as the dual BGG complex. We define cuspidal overconvergent Hilbert modular forms in Section 3 and show that the cohomology of the complex $\mathscr{C}^\bullet$ of cuspidal overconvergent Hilbert modular forms agrees with the rigid cohomology of the ordinary locus (Theorem 3.5). Moreover, we show that the slopes of $U_\mathfrak{p}$-operators are always greater or equal to the normalizing factor in Theorem 1 (Corollary 3.25). After this, we set up the spectral sequence that computes the rigid cohomology of the ordinary locus in Section 4. Entire Section 5 is devoted to give description of the cohomology of each GO-strata; this uses the earlier work [**TX13a**]. The last two sections each gives an approach to prove classicality; one unconditional but with some help from eigenvarieties; one is more straightforward but relying on some conjecture on partial Frobenius actions.

Acknowledgements. — We express our sincere gratitude to Kai-Wen Lan for his invaluable help on the compactification of Hilbert modular variety. We thank Ahmed Abbes, Matthew Emerton, Jianshu Li, Yifeng Liu, and Wei Zhang, for useful discussions.

We started working on this project when we were attending a workshop held at the Institute of Advance Study at Hongkong University of Science and Technology in December 2011. The hospitality of the institution and the well-organization provided us a great environment for brainstorming ideas. We especially thank the organizers Jianshu Li and Shou-wu Zhang, as well as the staff at IAS of HKUST.

Notation. — For a scheme X over a ring R and a ring homomorphism $R \to R'$, we use $X_{R'}$ to denote the base change $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$.

For a field F, we use Gal_F to denote its Galois group.

For a number field F, we use $\mathbb{A}_F$ to denote its ring of adeles, and $\mathbb{A}_F^\infty$ (resp. $\mathbb{A}_F^{\infty,p}$) to denote its finite adeles (resp. finite adeles away from places above p). When $F = \mathbb{Q}$, we suppress the subscript F from the notation. We put $\widehat{\mathbb{Z}}^{(p)} = \prod_{l \neq p} \mathbb{Z}_l$ and $\widehat{\mathcal{O}}_F^{(p)} = \prod_{\mathfrak{l} \nmid p} \mathcal{O}_\mathfrak{l}$. For each finite place $\mathfrak{p}$ of F, let $F_\mathfrak{p}$ denote the completion of F at $\mathfrak{p}$, $\mathcal{O}_\mathfrak{p}$ the ring of integers of $F_\mathfrak{p}$, and $k_\mathfrak{p}$ the residue field of $\mathcal{O}_\mathfrak{p}$. We put $d_\mathfrak{p} = [k_\mathfrak{p} : \mathbb{F}_p]$. Let $\varpi_\mathfrak{p}$ denote a uniformizer of $\mathcal{O}_\mathfrak{p}$, which we take to be the image of p when $\mathfrak{p}$ is unramified in $F/\mathbb{Q}$. We normalize the Artin map $\mathrm{Art}_F : F^\times \backslash \mathbb{A}_F^\times \to \mathrm{Gal}_F^{\mathrm{ab}}$ so that for each finite prime $\mathfrak{p}$, the element of $\mathbb{A}_F^\times$ whose $\mathfrak{p}$-component is $\varpi_\mathfrak{p}$ and other components are 1, is mapped to a geometric Frobenius at $\mathfrak{p}$.

We fix a totally real field F of degree $g > 1$ over $\mathbb{Q}$. Let $\mathfrak{d}_F$ be the different of F. Let Σ denote the set of places of F, and Σ_∞ the subset of all real places. We fix a prime number p which is unramified in F, and let Σ_p denote the set of places of F above p. We fix an isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$; this gives rise to a natural map $i_p : \Sigma_\infty \to \Sigma_p$ given by $\tau \mapsto \iota_p \circ \tau$. For each $\mathfrak{p} \in \Sigma_p$, we put $\Sigma_{\infty/\mathfrak{p}} = i_p^{-1}(\mathfrak{p})$.

For $\mathtt{S}$ an even subset of places of F, we use $B_\mathtt{S}$ to denote the quaternion algebra over F ramified at $\mathtt{S}$.

A *multiweight* is a tuple $(\underline{k}, w) = ((k_\tau)_{\tau \in \Sigma_\infty}, w) \in \mathbb{N}^{g+1}$ such that $w \geq k_\tau \geq 2$ and $w \equiv k_\tau \pmod 2$ for each τ. Let $\mathscr{A}_{(\underline{k},w)}$ denote the set of irreducible cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ whose archimedean component π_τ for each $\tau \in \Sigma_\infty$ is a discrete series of weight $k_\tau - 2$ with central character $x \mapsto x^{w-2}$. For such π, let $\rho_{\pi,l}$ denote the associated l-adic Galois representation, normalized so that $\det(\rho_{\pi,l})$ is $(1-w)$-power of the cyclotomic character.

For A an abelian scheme over a scheme S, we denote by $A^\vee$ the dual abelian scheme, by $\mathrm{Lie}(A/S)$ the Lie algebra of A, and by $\omega_{A/S}$ the module of *invariant* 1-*differential forms* of A relative to S. We sometimes omit S from the notation when the base is clear.

We shall frequently say Grothendieck group of modules of a ring R (resp. group ring of a topological group G); for that, we mean the Grothendieck group of *finitely generated* R-modules (resp. smooth admissible representations of G).

2. Preliminary on Hilbert Modular Varieties and Hilbert Modular Forms

In this section, we review the construction of the integral models of Hilbert modular varieties and their compactifications. We also recall the construction of the automorphic vector bundles, using the universal abelian varieties.

2.1. Shimura varieties for $\mathrm{GL}_{2,F}$. — Let G be the algebraic group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_{2,F})$ over $\mathbb{Q}$. Consider the homomorphism

$$h: \quad \mathbb{S}(\mathbb{R}) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m(\mathbb{R}) \cong \mathbb{C}^\times \longrightarrow G(\mathbb{R}) = \mathrm{GL}_2(F\otimes\mathbb{R})$$
$$a+\sqrt{-1}b \longmapsto \left(\begin{pmatrix} a & b \\ -b & a\end{pmatrix}, \dots, \begin{pmatrix} a & b \\ -b & a\end{pmatrix}\right).$$

The space of conjugacy classes of h under $G(\mathbb{R})$ has a structure of complex manifold, and is isomorphic to $(\mathfrak{h}^{\pm})^{\Sigma_\infty}$, where $\mathfrak{h}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ is the union of the upper half and lower half planes. For any open compact subgroup $K \subset G(\mathbb{A}^\infty) = \mathrm{GL}_2(\mathbb{A}_F^\infty)$, we have the Shimura variety $\mathrm{Sh}_K(G)$ with complex points

$$\mathrm{Sh}_K(G)(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathfrak{h}^{\pm})^{\Sigma_\infty} \times G(\mathbb{A}^\infty)/K$$

It is well known that $\mathrm{Sh}_K(G)$ has a canonical structure of quasi-projective variety defined over the reflex field $\mathbb{Q}$. For $g \in G(\mathbb{A}^\infty)$ and open compact subgroups $K, K' \subset G(\mathbb{A}^\infty)$ with $g^{-1}K'g \subset K$, there is a natural surjective map

$$[g] : \mathrm{Sh}_{K'}(G) \to \mathrm{Sh}_K(G) \tag{2.1.1}$$

whose effect on $\mathbb{C}$-points is given by $(z,h) \mapsto (z,hg)$. Thus we get a Hecke correspondence:

$$\begin{array}{ccccc} & & \mathrm{Sh}_{K\cap gKg^{-1}}(G) & & \\ & \swarrow & & \searrow^{[g]} & \\ \mathrm{Sh}_K(G) & & & & \mathrm{Sh}_K(G), \end{array} \tag{2.1.2}$$

where the left downward arrow is induced by the natural inclusion $K\cap gKg^{-1} \hookrightarrow K$, and the right downward one is given by $[g]$. Taking the project limit in K, we get a natural right action of $G(\mathbb{A}^\infty)$ on the projective limit $\mathrm{Sh}(G) := \varprojlim_K \mathrm{Sh}_K(G)$.

2.2. Automorphic Bundles. — Let $(\underline{k}, w)$ be a multiweight. We consider the algebraic representation of $G_{\mathbb{C}}$:

$$\rho^{(\underline{k},w)} : = \bigotimes_{\tau\in\Sigma_\infty}\left(\mathrm{Sym}^{k_\tau-2}(\check{\mathrm{St}}_\tau)\otimes \det_\tau^{-\frac{w-k_\tau}{2}}\right)$$

where $\check{\mathrm{St}}_\tau : G_{\mathbb{C}} \cong (\mathrm{GL}_{2,\mathbb{C}})^{\Sigma_\infty} \to \mathrm{GL}_{2,\mathbb{C}}$ is the *contragradient* of the projection onto the τ-th factor, and $\det_\tau$ is the projection onto τ-th factor composed with the determinant map. Consider the subgroup $Z_s = \mathrm{Ker}(\mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{N_{F/\mathbb{Q}}} \mathbb{G}_m)$ of the center $Z = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ of G; let G^c denote the quotient of G by Z_s. Then the representation $\rho^{(\underline{k},w)}$ factors through $G^c_{\mathbb{C}}$. Let L be a subfield of $\mathbb{C}$ that contains all the embeddings of F. The representation $\rho^{(\underline{k},w)}$ descends to a representation of G_L on an L-vector space $V^{(\underline{k},w)}$.

We say an open subgroup $K \subset G(\mathbb{A}^\infty)$ is *sufficiently small*, if the following two properties are satisfied:

1. The quotient $(g^{-1}Kg \cap \mathrm{GL}_2(F))/(gKg^{-1}\cap F^\times)$ does not have non-trivial elements of finite order for all $g \in G(\mathbb{A}^\infty)$.
2. $N_{F/\mathbb{Q}}(K\cap F^\times)^{w-2} = 1$.

If K is sufficiently small, it follows from [**Mil90a**, Chap. III 3.3] that $\rho^{(\underline{k},w)}$ gives rise to an algebraic vector bundle $\mathscr{F}^{(\underline{k},w)}$ on $\mathrm{Sh}_K(G)$ equipped with an integrable connection

$$\nabla : \mathscr{F}^{(\underline{k},w)} \to \mathscr{F}^{(\underline{k},w)} \otimes \Omega^1_{\mathrm{Sh}_K(G)_L}.$$

The theory of automorphic bundles also allows us to define an invertible sheaf on $\mathrm{Sh}_K(G)$ for K sufficiently small as follows. Consider the compact dual $(\mathbb{P}^1_{\mathbb{C}})^{\Sigma_\infty}$ of the Hermitian symmetric domain $(\mathfrak{h}^{\pm})^{\Sigma}_{\infty}$. It has a natural action by $G_{\mathbb{C}} = (\mathrm{GL}_{2,\mathbb{C}})^{\Sigma_\infty}$. Let ω be the dual of the tautological quotient bundle on $\mathbb{P}^1_{\mathbb{C}}$. Then the line bundle ω has a natural $\mathrm{GL}_{2,\mathbb{C}}$-equivariant action. We define

$$\underline{\omega}^{(\underline{k},w)} \colon = \bigotimes_{\tau\in\Sigma_\infty} \mathrm{pr}_\tau^*(\omega^{\otimes(k_\tau-2)} \otimes \det{}^{-\frac{w-k_\tau}{2}}) \tag{2.2.1}$$

and a $G_{\mathbb{C}}$-equivariant action on $\underline{\omega}^{(\underline{k},w)}$ as follows. For each $\tau \in \Sigma_\infty$, the action of $G_{\mathbb{C}}$ on $\mathrm{pr}_\tau^*(\omega^{\otimes(k_\tau-2)} \otimes \det^{-\frac{w-k_\tau}{2}})$ factors through the τ-th copy of $\mathrm{GL}_{2,\mathbb{C}}$, which in turn acts as the product of $\det^{-\frac{w-k_\tau}{2}}$ and the $(k_\tau-2)$-th power of the natural action on ω. One checks easily that the action of $G_{\mathbb{C}}$ on $\omega^{(\underline{k},w)}$ factors through $G^c_{\mathbb{C}}$, and thus $\underline{\omega}^{(\underline{k},w)}$ descends to an invertible sheaf on $\mathrm{Sh}_K(G)$ for K sufficiently small by [**Mil90a**]. As usual, the invertible sheaf $\underline{\omega}^{(\underline{k},w)}$ on $\mathrm{Sh}_K(G)$ has a canonical model over L.

We define the space of holomorphic Hilbert modular forms of level K with coefficients in L to be

$$M_{(\underline{k},w)}(K,L) \colon = H^0(\mathrm{Sh}_K(G)_L, \underline{\omega}^{(\underline{k},w)} \otimes \Omega^g_{\mathrm{Sh}_K(G)}). \tag{2.2.2}$$

Note here that the canonical bundle $\Omega^g_{\mathrm{Sh}_K(G)}$ accounts for a parallel weight two automorphic line bundle; so our definition is equivalent to the usual notion of holomorphic Hilbert modular forms. (In fact, we will see later that our formulation is more natural in various ways.)

Explicitly, an element of $M_{(\underline{k},w)}(K,\mathbb{C})$ is a function $f(z,g)$ on $((\mathfrak{h}^{\pm})^{\Sigma_\infty}, G(\mathbb{A}^\infty))$ such that

1. $f(z,g)$ is holomorphic in z and locally constant in g; and
2. one has $f(z,gk) = f(z,g)$ for any $k\in K$, and

$$f(\gamma(z),\gamma g) = \Big(\prod_{\tau\in\Sigma_\infty} \frac{(c_\tau z_\tau + d_\tau)^{k_\tau}}{\det(\gamma_\tau)^{\frac{w+k_\tau-2}{2}}}\Big) f(z,g),$$

   where $\gamma \in G(\mathbb{Q})$, and $\gamma_\tau = \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau\end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ is the image of γ via $G(\mathbb{Q}) \hookrightarrow \mathrm{GL}_2(F\otimes\mathbb{R}) \xrightarrow{\mathrm{pr}_\tau} \mathrm{GL}_2(\mathbb{R})$, and $\gamma(z) = \big(\frac{a_\tau z_\tau + b_\tau}{c_\tau z_\tau + d_\tau}\big)_{\tau\in\Sigma_\infty}$.

We denote by $S_{(\underline{k},w)}(K,L) \subset M_{(\underline{k},w)}(K,L)$ be the subspace of cusp forms. For each $g\in G(\mathbb{A}^\infty)$ and open compact subgroups $K,K' \subset G(\mathbb{A}^\infty)$ with $g^{-1}K'g \subset K$, by construction, there exists a natural isomorphism of coherent sheaves on $\mathrm{Sh}_{K'}(G)$:

$$[g]^*(\underline{\omega}^{(\underline{k},w)}) \xrightarrow{\sim} \underline{\omega}^{(\underline{k},w)}.$$

Together with the map (2.1.1), one deduces a map $S_{(\underline{k},w)}(K,L) \to S_{(\underline{k},w)}(K',L)$. Passing to limit, one obtains a natural left action of $G(\mathbb{A}^\infty)$ on $S_{(\underline{k},w)}(L) = \varinjlim_K S_{(\underline{k},w)}(K,L)$, such that $S_{(\underline{k},w)}(K,L)$ is identified with the invariants of $S_{(\underline{k},w)}(L)$ under K. Let $\mathscr{A}_{(\underline{k},w)}$ be the set of cuspidal automorphic representations $\pi = \pi_f \otimes \pi_\infty$ of $\mathrm{GL}_2(\mathbb{A}_F)$, such that each archimedean component π_τ of π for $\tau\in\Sigma_\infty$ is the discrete series of weight k_τ and central character $x \mapsto x^{w-2}$. Then we have canonical decompositions

$$S_{(\underline{k},w)}(\mathbb{C}) = \bigoplus_{\pi=\pi^\infty\otimes\pi_\infty\in\mathscr{A}_{(\underline{k},w)}} \pi^\infty \quad \text{and} \quad S_{(\underline{k},w)}(K,\mathbb{C}) = \bigoplus_{\pi=\pi^\infty\otimes\pi_\infty\in\mathscr{A}_{(\underline{k},w)}} (\pi^\infty)^K.$$

where π^∞ denotes the finite part of π.

2.3. Moduli interpretation and integral models. — Recall that p is a rational prime unramified in F. We consider level structures of the type $K = K^pK_p$, where $K^p \subset G(\mathbb{A}^{\infty,p})$ is an open compact subgroup, and K_p is hyperspecial, i.e. $K_p \simeq \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$. For every finite place v, we denote by $K_v \subset \mathrm{GL}_2(F_v)$ the v-component of K. We will use the moduli interpretation to define integral models of $\mathrm{Sh}_K(G)$, for sufficiently small K^p.

We start with a more transparent description of $\mathrm{Sh}_K(G)(\mathbb{C})$. The determinant map $\det : G \to \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ induces a bijection between the set of geometric connected components of $\mathrm{Sh}_K(G)$ and the double coset space

$$cl_F^+(K) \colon = F_+^\times \backslash \mathbb{A}_F^{\infty,\times} / \det(K),$$

where $F_+^\times$ denotes the subgroup of $F^\times$ of totally positive elements. Since $\det(K) \subseteq \prod_{v \nmid \infty} \mathcal{O}_{F_v}^\times$, there is a natural surjective map $cl_F^+(K) \to cl_F^+$, where cl_F^+ is the strict ideal class group of F. The preimage of each ideal class $[\mathfrak{c}]$ is a torsor under the group $I := \widehat{\mathcal{O}}_F^\times / \det(K)\mathcal{O}_{F,+}^\times$.

We fix fractional ideals $\mathfrak{c}_1, \cdots, \mathfrak{c}_{h_F^+}$ coprime to p, which form a set of representatives of cl_F^+. For each $\mathfrak{c} = \mathfrak{c}_j$, we choose a subset $[\mathfrak{c}]_K = \{g_i \,|\, i \in I\} \subset G(\mathbb{A}^\infty)$ such that the fractional ideal associated to every $\det(g_i)$ is $\mathfrak{c}$ and $\{\det(g_i) \,|\, i \in I\}$ is a set of representatives of the pre-image of $\mathfrak{c}$ in $cl_F^+(K)$. By the strong approximation theorem for $\mathrm{SL}_{2,F}$, we have $G(\mathbb{A}^\infty) = \coprod_{\mathfrak{c} \in cl_F^+} \coprod_{g_i \in [\mathfrak{c}]_K} G(\mathbb{Q})^+ g_i K$, where $G(\mathbb{Q})^+$ is the subgroup of $G(\mathbb{Q})$ with totally positive determinant. This gives rise to a decomposition

$$\mathrm{Sh}_K(G)(\mathbb{C}) = G(\mathbb{Q})^+ \backslash \mathfrak{h}^{\Sigma_\infty} \times G(\mathbb{A}^\infty)/K = \coprod_{\mathfrak{c} \in cl_F^+} \mathrm{Sh}_K^{\mathfrak{c}}(G)(\mathbb{C}),$$

$$\text{where} \quad \mathrm{Sh}_K^{\mathfrak{c}}(G)(\mathbb{C}) = \coprod_{g_i \in [\mathfrak{c}]_K} \Gamma(g_i, K) \backslash \mathfrak{h}^{\Sigma_\infty} \quad \text{with } \Gamma(g_i, K) = g_i K g_i^{-1} \cap G(\mathbb{Q})^+. \tag{2.3.1}$$

We note that $\mathrm{Sh}_K^{\mathfrak{c}}(G)$ does not depend on the choice of the subset $[\mathfrak{c}]_K = \{g_i : i \in I\}$, and descends to an algebraic variety defined over $\mathbb{Q}$. We will interpret $\mathrm{Sh}_K^{\mathfrak{c}}(G)$ as a moduli space as follows.

Assume that $\mathfrak{c}$ is coprime to p. Let $\mathfrak{c}^+$ be the cone of totally positive elements of $\mathfrak{c}$. Let S be a connected locally noetherian $\mathbb{Z}_{(p)}$-scheme.

– A *Hilbert-Blumenthal abelian variety* (*HBAV* for short) (A, ι) over S is an abelian variety A/S of dimension $[F : \mathbb{Q}]$ together with a homomorphism $\iota : \mathcal{O}_F \to \mathrm{End}_S(A)$ such that $\mathrm{Lie}(A)$ is a locally free $(\mathcal{O}_S \otimes_{\mathbb{Z}} \mathcal{O}_F)$-module of rank 1.
– If (A, ι) is an HBAV over S, then $A^\vee$ has a natural action by $\mathcal{O}_F$. Let $\mathrm{Hom}_{\mathcal{O}_F}^{\mathrm{Sym}}(A, A^\vee)$ denote the group of symmetric homomorphisms of A to $A^\vee$, and $\mathrm{Hom}_{\mathcal{O}_F}^{\mathrm{Sym}}(A, A^\vee)^+$ be the cone of polarization. A $\mathfrak{c}$-*polarization* on A is an $\mathcal{O}_F$-linear isomorphism

$$\lambda : (\mathfrak{c}, \mathfrak{c}^+) \xrightarrow{\sim} \big(\mathrm{Hom}_{\mathcal{O}_F}^{\mathrm{Sym}}(A, A^\vee), \mathrm{Hom}_{\mathcal{O}_F}^{\mathrm{Sym}}(A, A^\vee)^+\big)$$

preserving the positive cones on both sides; in particular, λ induces an isomorphism of HBAVs: $A \otimes_{\mathcal{O}_F} \mathfrak{c} \simeq A^\vee$.
– We define first the level structure for $K = K(N)$, the principal subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ modulo an integer N coprime to p. A *principal level-N (or a level-$K(N)^p$) structure* on a $\mathfrak{c}$-polarized HBAV (A, ι, λ) is an $\mathcal{O}_F$-linear isomorphism of finite étale group schemes over S

$$\alpha_N : (\mathcal{O}_F/N)^{\oplus 2} \xrightarrow{\sim} A[N].$$

Note that there exists a natural $\mathcal{O}_F$-pairing $A[N] \times A^\vee[N] \to \mu_N \otimes_{\mathbb{Z}} \mathfrak{d}_F^{-1}$. Its composition with $1 \otimes \lambda$ gives an $\mathcal{O}_F$-linear alternating pairing $A[N] \times A[N] \to \mu_N \otimes_{\mathbb{Z}} \mathfrak{c}^*$. Hence, α_N determines an isomorphism

$$\nu(\alpha_N) : \mathcal{O}_F/N\mathcal{O}_F = \wedge^2_{\mathcal{O}_F}(\mathcal{O}_F/N)^{\oplus 2} \xrightarrow{\sim} \wedge^2_{\mathcal{O}_F} A[N] \xrightarrow{\sim} \mu_N \otimes_{\mathbb{Z}} \mathfrak{c}^*.$$

For a general open compact subgroup $K \subseteq \mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ with $K_p = \mathrm{GL}_2(\mathcal{O}_F \otimes \widehat{\mathbb{Z}}_p)$, we define a level-$K^p$ structure on (A, ι, λ) as follows. Choose an integer N coprime to p such that $K(N) \subseteq K$, and a geometric point $\overline{s}$ of S. The finite group $\mathrm{GL}_2(\mathcal{O}_F/N)$ acts naturally on the set of principal level-N structures $(\alpha_{N,\overline{s}}, \nu(\alpha_{N,\overline{s}}))$ of $A_{\overline{s}}$ by putting

$$g : (\alpha_{N,\overline{s}}, \nu(\alpha_{N,\overline{s}})) \mapsto (\alpha_{N,\overline{s}} \circ g, \det(g)\nu(\alpha_{N,\overline{s}})).$$

Then a *level-K^p structure* α_{K^p} on (A, ι, λ) is an $\pi_1(S, \overline{s})$-invariant $K/K(N)$-orbit of the pairs $(\alpha_{N,\overline{s}}, \nu(\alpha_{N,\overline{s}}))$. This definition does not depend on the choice of N and $\overline{s}$.

We consider the moduli problem which associates to each connected locally noetherian $\mathbb{Z}_{(p)}$-schemes S, the set of isomorphism classes of quadruples $(A, \iota, \lambda, \alpha_{K^p})$ as above. If K^p is sufficiently small so that any $(A, \iota, \lambda, \alpha_{K^p})$ does not admit non-trivial automorphisms, then this moduli problem is representable by a smooth and quasi-projective $\mathbb{Z}_{(p)}$-scheme $\mathcal{M}^{\mathfrak{c}}_K$ [**Ra78, Ch90**]. After choosing a primitive N-th root of unity ζ_N for some integer N coprime to p such that $K(N) \subseteq K$, the set of geometric connected components of $\mathcal{M}^{\mathfrak{c}}_K$ is in natural bijection with [**Ch90**, 2.4]

$$\mathrm{Isom}(\widehat{\mathcal{O}}_F, \widehat{\mathcal{O}}_F \otimes \mathfrak{c}^*)/\det(K) \simeq \prod_{v|N} \mathcal{O}^\times_{F_v}/\det(K_v).$$

Let $\mathcal{O}^\times_{F,+}$ be the group of totally positive units of $\mathcal{O}_F$. It acts on $\mathcal{M}^{\mathfrak{c}}_K$ as follows. For $\epsilon \in \mathcal{O}^\times_{F,+}$ and an S-point $(A, \iota, \lambda, \alpha_{K^p})$, we put $\epsilon \cdot (A, \iota, \lambda, \alpha_{K^p}) = (A, \iota, \iota(\epsilon) \circ \lambda, \alpha_{K^p})$. We point out that this action will take $\nu(\alpha_{N,\bar{s}})$ to $\epsilon\nu(\alpha_{N,\bar{s}})$. We will denote by $(A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$ the associated $\mathcal{O}^\times_{F,+}$-orbit of $(A, \iota, \lambda, \alpha_{K^p})$. The subgroup $(K \cap \mathcal{O}^\times_F)^2$ acts trivially on $\mathcal{M}^{\mathfrak{c}}_K$, where $\mathcal{O}^\times_F$ is considered as a subgroup of the center of $\mathrm{GL}_2(\mathbb{A}^\infty)$. Indeed, if $\epsilon = u^2$ with $u \in K \cap \mathcal{O}^\times_F$, the endomorphism $\iota(u) : A \to A$ induces an isomorphism of quadruples $(A, \iota, \lambda, \alpha_{K^p}) \simeq (A, \iota, \iota(\epsilon) \circ \lambda, \alpha_{K^p})$. Hence, the action of $\mathcal{O}^\times_{F,+}$ on $\mathcal{M}^{\mathfrak{c}}_K$ factors through the finite quotient $\mathcal{O}^\times_{F,+}/(K \cap \mathcal{O}^\times_F)^2$. The equivalent classes of the set of geometric connected components of $\mathcal{M}^{\mathfrak{c}}_K$ under the induced action of $\mathcal{O}^\times_{F,+}/(K \cap \mathcal{O}^\times_F)^2$ is in bijection with $\widehat{\mathcal{O}}^\times_F/\det(K)\mathcal{O}^\times_{F,+}$, and the stabilizer of each geometric connected component is $\big(\det(K) \cap \mathcal{O}^\times_{F,+}\big)/(K \cap \mathcal{O}^\times_F)^2$.

Proposition 2.4. — *There exists an isomorphism between the quotient of $\mathcal{M}^{\mathfrak{c}}_K(\mathbb{C})$ by $\mathcal{O}^\times_{F,+}/(K \cap \mathcal{O}^\times_F)^2$ and $\mathrm{Sh}^{\mathfrak{c}}_K(G)(\mathbb{C})$. In other words, $\mathrm{Sh}^{\mathfrak{c}}_K(G)(\mathbb{C})$ is identified with the coarse moduli space over $\mathbb{C}$ of the quadruples $(A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$. Moreover, if $\det(K) \cap \mathcal{O}^\times_{F,+} = (K \cap \mathcal{O}^\times_F)^2$, then the quotient map $\mathcal{M}^{\mathfrak{c}}_K \to \mathrm{Sh}^{\mathfrak{c}}_K(G)(\mathbb{C})$ induces an isomorphism between any geometric connected component of $\mathcal{M}^{\mathfrak{c}}_K$ with its image.*

Proof. — We fix an idèle $a \in \mathbb{A}^{\infty,\times}_F$ whose associated fractional ideal is $\mathfrak{c}$. Let $\Delta \subseteq \widehat{\mathcal{O}}^\times_F$ be a complete subset of representatives of $\widehat{\mathcal{O}}^\times_F/\det(K)$, and let $I \subset \Delta$ be a subset of representatives of $\widehat{\mathcal{O}}^\times_F/\det(K)\mathcal{O}^\times_{F,+}$. We put $g_\delta = \begin{pmatrix} \delta a & 0 \\ 0 & 1 \end{pmatrix}$ for $\delta \in \Delta$, and $\Gamma(g, K) = gKg^{-1} \cap G(\mathbb{Q})^+$ and $\Gamma^1(g, K) = \Gamma(g, K) \cap \mathrm{SL}_2(F)$. Then it is well known that

$$\mathcal{M}^{\mathfrak{c}}_K(\mathbb{C}) = \coprod_{\delta \in \Delta} \Gamma^1(g_\delta, K)\backslash \mathfrak{h}^{\Sigma_\infty}.$$

The case for $K = K(N)$ is proved in [**Ra78**] or [**Hi**, 4.1.3], and the general case is similar. For $\epsilon \in \mathcal{O}^\times_{F,+}$, it sends a point $\Gamma^1(g_\delta, K)z$ in $\Gamma^1(g_\delta, K)\backslash \mathfrak{h}^{\Sigma_\infty}$ to $\Gamma^1(g_{\delta\epsilon}, K)\epsilon z$. Hence the quotient of $\mathcal{M}^{\mathfrak{c}}_K(\mathbb{C})$ is isomorphic to

$$\coprod_{\delta \in I} \Big(\big(\Gamma^1(g_\delta, K)\backslash \mathfrak{h}^{\Sigma_\infty}\big)/(\det(K) \cap \mathcal{O}^\times_{F,+})/(K \cap \mathcal{O}_F)^2 \Big).$$

On the other hand, taking $[\mathfrak{c}]_K = \{g_\delta, \delta \in I\}$, one gets (2.3.1): $\mathrm{Sh}^{\mathfrak{c}}_K(G)(\mathbb{C}) = \coprod_{\delta \in I} \Gamma(g_\delta, K)\backslash \mathfrak{h}^{\Sigma_\infty}$. Note that for each $\delta \in I$, $\Gamma(g_\delta, K)\backslash \mathfrak{h}^{\Sigma_\infty}$ is identified with the natural quotient of

$\Gamma^1(g_\delta, K)\backslash\mathfrak{h}^{\Sigma_\infty}$ by the group

$$F^\times\Gamma(g_\delta, K)/\Gamma^1(g_\delta, K)F^\times \simeq \Gamma(g_\delta, K)/(F^\times \cap \Gamma(g_\delta, K))\Gamma^1(g_\delta, K).$$

By the strong approximation for $\mathrm{SL}_{2,F}$, one sees that $\det : \Gamma(g_\delta, K) \to \det(K) \cap \mathcal{O}_{F,+}^\times$ is surjective. Hence, the group above is isomorphic to $(\det(K) \cap \mathcal{O}_{F,+}^\times)/(K \cap \mathcal{O}_{F,+}^\times)^2$. The Proposition follows immediately. □

We define $\mathbf{Sh}_K^{\mathfrak{c}}(G)$ to be the quotient of $\mathcal{M}_K^{\mathfrak{c}}$ by the action of the finite group $\det(K) \cap \mathcal{O}_{F,+}^\times/(K \cap \mathcal{O}_F^\times)^2$, and we put $\mathbf{Sh}_K(G) = \coprod_{\mathfrak{c} \in cl^+(F)} \mathbf{Sh}_K^{\mathfrak{c}}(G)$. In general, this is just a coarse moduli space that parametrizes the quadruples $(A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$. However, we have the following:

Lemma 2.5. — *For any open compact subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$, there exists an open compact normal subgroup $K'^p \subseteq K^p$ of finite index, such that $\det(K'^pK_p) \cap \mathcal{O}_{F,+}^\times = (K'^pK_p \cap \mathcal{O}_F^\times)^2$.*

Proof. — By a theorem of Chevalley (see for instance [**Ta03**, Lemma 2.1]), every finite index subgroup of $\mathcal{O}_F^\times$ contains a subgroup of the form $U \cap \widehat{\mathcal{O}}_F^\times$, where $U \subseteq \hat{\mathcal{O}}_F^\times$ is an open compact subgroup with $U_v = \mathcal{O}_{F_v}^\times$ for all $v|p$. Therefore, one can choose such an open compact $U \subseteq \det(K)$ such that $U \cap (\det(K) \cap \mathcal{O}_F^{\times,+}) = U \cap (K \cap \mathcal{O}_F^\times)^2$. Let $K'^p \subseteq K^p$ denotes the inverse image of U^p via determinant map. Then it is easy to check that this choice of K'^p answers the question. □

Remark 2.6. — In general, $\det(K)\cap\mathcal{O}_{F,+}^\times/(K\cap\mathcal{O}_F^\times)^2$ is non-trivial even for K^p sufficiently small. For instance, if $K = K(N)$ for some integer N coprime to p, then $K \cap \mathcal{O}_F^\times$ is the subgroup of units congruent to 1 modulo N, and $\det(K) \cap \mathcal{O}_{F,+}^\times$ is subgroup of $K \cap \mathcal{O}_F^\times$ of positive elements. By the theorem of Chevalley cited in the proof of the Lemma, we have $\det(K) \cap \mathcal{O}_{F,+}^\times = K \cap \mathcal{O}_F^\times$ for N sufficiently large, and hence $\det(K) \cap \mathcal{O}_{F,+}^\times/(K \cap \mathcal{O}_F^\times)^2 \simeq (\mathbb{Z}/2\mathbb{Z})^{[F:\mathbb{Q}]-1}$.

From now on, we always make the following

Hypothesis 2.7. — K^p is sufficiently small and $\det(K) \cap \mathcal{O}_{F,+}^\times = (K \cap \mathcal{O}_F^\times)^2$.

By Lemma 2.5, this hypothesis is always valid up to replacing K^p by an open compact subgroup. Under this assumption, Proposition 2.4 shows that each geometric connected component is identified with a certain geometric connected component of $\mathcal{M}_K^{\mathfrak{c}}$. Therefore, $\mathbf{Sh}_K(G)$ is quasi-projective and smooth over $\mathbb{Z}_{(p)}$. We can also talk about the universal family of HBAV over $\mathbf{Sh}_K(G)$.

Remark 2.8. — In the construction of $\mathbf{Sh}_K(G)$, the set of representatives $\{\mathfrak{c}_1, \cdots, \mathfrak{c}_{h_F^+}\}$ of cl_F^+ are assumed to be coprime to p. This assumption is used to prove the smoothness of each $\mathcal{M}_K^{\mathfrak{c}_i}$, hence that of $\mathbf{Sh}_K^{\mathfrak{c}_i}(G)$ using deformation theory. However, dropping this assumption will not cause any problems in practice. Suppose we are given a quadruple $(A, \iota, \lambda, \alpha_{K^p})$ over a connected locally noetherian $\mathbb{Z}_{(p)}$-scheme S, where $\lambda : \mathfrak{q} \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{O}_F}^{\mathrm{Sym}}(A, A^\vee)$ is an isomorphism preserving positivity for a not necessarily prime-to-p fractional ideal $\mathfrak{q}$. Then there exists a unique representative $\mathfrak{c}_i$ and a $\xi \in F_+^\times$ such that multiplication by ξ defines an isomorphism $\xi : \mathfrak{c}_i \xrightarrow{\simeq} \mathfrak{q}$. We put $\lambda' = \xi \circ \lambda$. Let $(\alpha_{N,\bar{s}}, \nu(\alpha_{N,\bar{s}}))$ be a representative of isomorphisms in the level-K^p structure α_{K^p} for some integer N coprime to p with $K(N) \subseteq K$. We define α'_{K^p} to be the $K/K(N)$-orbit of the pairs $(\alpha_{N,\bar{s}}, \xi \cdot \nu(\alpha_{N,\bar{s}}))$, where $\xi \cdot \nu(\alpha_{N,\bar{s}})$ is the composite isomorphism

$$\mathcal{O}_F/N\mathcal{O}_F \xrightarrow{\nu(\alpha_{N,\bar{s}})} \mu_{N,\bar{s}} \otimes_{\mathbb{Z}} \mathfrak{q}^* \xrightarrow{\xi} \mu_N \otimes_{\mathbb{Z}} \mathfrak{c}_i^*.$$

We then get a new quadruple $(A, \iota, \lambda', \alpha'_{K^p})$. Since ξ is well determined up to $\mathcal{O}_{F,+}^{\times}$, the $\mathcal{O}_{F,+}^{\times}$-orbit $(A, \iota, \bar{\lambda}', \bar{\alpha}'_{K^p})$ well defines an S-point in $\mathbf{Sh}_K(G)$. By abuse of notation, we also use $(A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$ to denote this point.

2.9. Hecke actions on $\mathbf{Sh}_K(G)$. — Suppose we are given $g \in G(\mathbb{A}^{\infty,p})$, and open compact subgroups $K^p, K'^p \subset G(\mathbb{A}^{\infty,p})$ with $g^{-1}K'^p g \subseteq K^p$. We let $K = K^p K_p$, $K' = K'^p K'_p$ with $K_p = K'_p = \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. Then we have a finite étale map

$$[g] : \mathbf{Sh}_{K'}(G) \to \mathbf{Sh}_K(G) \tag{2.9.1}$$

that extends the Hecke action (2.1.1). If K^p and g are both contained in $\mathrm{GL}_2(\widehat{\mathcal{O}}_F^{(p)})$, the morphism $[g]$ is given by $(A, \iota, \bar{\lambda}, \bar{\alpha}_{K'^p}) \mapsto (A, \iota, \bar{\lambda}, [\bar{\alpha}_{K'^p} \circ g]_K)$, where $[\bar{\alpha}_{K'^g} \circ g]_K$ denotes the K^p-level structure associated to $\bar{\alpha}_{K'^p} \circ g$. To define $[g]$ in the general case, it is more natural to use the rational version of the moduli interpretation of $\mathcal{M}_K$ as in [**La13a**, 6.4.3], i.e. we consider $\mathcal{M}_K$ as the classifying space of certain isogenies classes of HBAVs instead of the classifying space of isomorphism classes of HBAV. For more details on these two types of moduli interpretation for $\mathcal{M}_K$ and their equivalence, we refer the reader to [**La13a**, Section 1.4] and [**Hi**, Section 4.2.1].

2.10. Compactifications. — Let $K = K^p K_p \subset G(\mathbb{A}^{\infty})$ be an open compact subgroup with K_p hyperspecial and satisfying Hypothesis 2.7. We recall some results on the arithmetic toroidal compactification of $\mathbf{Sh}_K(G)$. For more details, the reader may refer to [**Ra78, Ch90**] and more recently [**La13a**, Chap. VI].

By choosing suitable admissible rational polyhedral cone decomposition data for $\mathbf{Sh}_K(G)$, one can construct arithmetic toroidal compactifications $\mathbf{Sh}_K^{\mathrm{tor}}(G)$ satisfying the following conditions.

1. The schemes $\mathbf{Sh}_K^{\mathrm{tor}}(G)$ are projective and smooth over $\mathbb{Z}_{(p)}$.
2. There exists natural an open immersion $\mathbf{Sh}_K(G) \hookrightarrow \mathbf{Sh}_K^{\mathrm{tor}}(G)$ such that the boundary $\mathbf{Sh}_K^{\mathrm{tor}}(G) - \mathbf{Sh}_K(G)$ is a relative simple normal crossing Cartier divisor of $\mathbf{Sh}_K^{\mathrm{tor}}(G)$ with respect to the base.
3. There exists a polarized semi-abelian scheme $\mathcal{A}^{\mathrm{sa}}$ over $\mathbf{Sh}_K^{\mathrm{tor}}(G)$ equipped with an action of $\mathcal{O}_F$ and a K^p-level structure, which extends the universal abelian scheme $\mathcal{A}$ on $\mathbf{Sh}_K(G)$ and degenerates to torus at cusps.
4. Suppose we are given an element $g \in G(\mathbb{A}^{\infty,p})$, and open compact subgroups $K^p, K'^p \subset G(\mathbb{A}^{\infty,p})$ with $g^{-1}K'^p g \subseteq K^p$. We put $K = K^p K_p$, $K' = K'^p K'_p$ with $K_p = K'_p = \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. Then by choosing compatible rational polyhedral cone decomposition data for $\mathbf{Sh}_K(G)$ and $\mathbf{Sh}_{K'}(G)$, we have a proper surjective morphism [**La13a**, 6.4.3.4]:

$$[g]^{\mathrm{tor}} : \mathbf{Sh}_{K'}^{\mathrm{tor}}(G) \to \mathbf{Sh}_K^{\mathrm{tor}}(G), \tag{2.10.1}$$

whose restriction to $\mathbf{Sh}_{K'}(G)$ is (2.9.1) defined by the Hecke action of g. Moreover, $[g]^{\mathrm{tor}}$ is log-étale if we equip with $\mathbf{Sh}_{K'}^{\mathrm{tor}}(G)$ and $\mathbf{Sh}_K^{\mathrm{tor}}(G)$ the canonical log-structures given by their toroidal boundaries. Each double coset $K^p g K^p$ with $g \in \mathrm{GL}_2(\mathbb{A}^{\infty,p})$ defines an extended Hecke correspondence

$$\begin{array}{ccc} & \mathbf{Sh}^{\mathrm{tor}}_{K \cap gKg^{-1}}(G) & \\ {}^{[1]^{\mathrm{tor}}}\swarrow & & \searrow^{[g]^{\mathrm{tor}}} \\ \mathbf{Sh}_K^{\mathrm{tor}}(G) & & \mathbf{Sh}_K^{\mathrm{tor}}(G), \end{array} \tag{2.10.2}$$

which extends (2.1.2).

We put $\underline{\omega} = e^*(\Omega^1_{\mathcal{A}^{\mathrm{sa}}/\mathbf{Sh}_K^{\mathrm{tor}}(G)})$, where $e : \mathbf{Sh}_K^{\mathrm{tor}}(G) \to \mathcal{A}^{\mathrm{sa}}$ denotes the unit section. It is an $(\mathcal{O}_{\mathbf{Sh}_K^{\mathrm{tor}}} \otimes_{\mathbb{Z}} \mathcal{O}_F)$-module locally free of rank 1, and it extends the sheaf of invariant 1-differentials of $\mathcal{A}$ over $\mathbf{Sh}_K(G)$. We define the *Hodge line bundle* to be $\det(\underline{\omega}) = \bigwedge^g_{\mathcal{O}_{\mathbf{Sh}_K^{\mathrm{tor}}(G)}} \underline{\omega}$.

Following [**Ch90**] and [**La13a**, Section 7.2], we put

$$\mathbf{Sh}^*_K(G) = \mathrm{Proj}(\bigoplus_{n\geq 0}\Gamma(\mathbf{Sh}^{\mathrm{tor}}_K(G), \det(\underline{\omega})^{\otimes n})).$$

This is a normal and projective scheme over $\mathbb{Z}_{(p)}$, and $\det(\underline{\omega})$ descends to an ample line bundle on $\mathbf{Sh}^*_K(G)$. Moreover, the inclusion $\mathbf{Sh}_K(G) \hookrightarrow \mathbf{Sh}^{\mathrm{tor}}_K(G)$ induces an inclusion $\mathbf{Sh}_K(G) \hookrightarrow \mathbf{Sh}^*_K(G)$. Although $\mathbf{Sh}^{\mathrm{tor}}_K(G)$ depends on the choice of certain cone decompositions, $\mathbf{Sh}^*_K(G)$ is canonically determined by $\mathbf{Sh}_K(G)$. We call $\mathbf{Sh}^*_K(G)$ the *minimal compactification* of $\mathbf{Sh}_K(G)$. The boundary $\mathbf{Sh}^*_K(G) - \mathbf{Sh}_K(G)$ is finite flat over $\mathbb{Z}_{(p)}$, and its connected components are indexed by the cusps of $\mathbf{Sh}_K(G)$.

2.11. de Rham cohomology. — Let F^{Gal} be the Galois closure of F in $\overline{\mathbb{Q}}$. Let R be an $\mathcal{O}_{F^{\mathrm{Gal}},(p)}$-algebra. In practice, we will need the cases where R equals to $\mathbb{C}$, a finite field k of characteristic p sufficiently large, or a finite extension $L/\mathbb{Q}_p$ that contains all the embeddings of F into $\overline{\mathbb{Q}}_p$, or the ring of integers of such L. Let Σ_R be the set of the g distinct algebra homomorphisms from $\mathcal{O}_F$ to R. Hence, we have $\Sigma_\infty = \Sigma_R$ in this notation. To simplify the notation, we put $\mathbf{Sh}_{K,R} := \mathbf{Sh}_K(G)_R$ and $\mathbf{Sh}^{\mathrm{tor}}_{K,R} := \mathbf{Sh}^{\mathrm{tor}}_K(G)_R$, and we write $\mathbf{Sh}_K(\mathbb{C})$ and $\mathbf{Sh}^{\mathrm{tor}}_K(\mathbb{C})$ for the associated complex manifolds respectively. For a coherent $(\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}\otimes_{\mathbb{Z}}\mathcal{O}_F)$-module M, we denote by $M = \bigoplus_{\tau\in\Sigma_R} M_\tau$ the canonical decomposition, where M_τ is the direct summund on which $\mathcal{O}_F$ acts via $\tau : \mathcal{O}_F \to R \to \mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}$.

Let D denote the boundary $\mathbf{Sh}^{\mathrm{tor}}_{K,R} - \mathbf{Sh}_{K,R}$, and $\Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D})$ be the sheaf of 1-differentials on $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$ over $\mathrm{Spec}(R)$ with logarithmic poles along the relative normal crossing Cartier divisor D. Using a toroidal compactification of the semi-abelian scheme $\mathcal{A}^{\mathrm{sa}}$ on $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$, there exists a unique $(\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}\otimes\mathcal{O}_F)$-module $\mathcal{H}^1$ locally free of rank 2 satisfying the following properties [**La11**, 2.15, 6.9]:

1. The restriction of $\mathcal{H}^1$ to $\mathbf{Sh}_{K,R}$ is the relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathbf{Sh}_{K,R})$ of the universal abelian scheme $\mathcal{A}$. Actually, $\mathcal{H}^1$ is called the *canonical extension* of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathbf{Sh}_{K,R})$ in [**La11**, 6.9].
2. There exists a canonical $\mathcal{O}_F$-equivariant Hodge filtration
$$0 \to \underline{\omega} \to \mathcal{H}^1 \to \mathrm{Lie}((\mathcal{A}^{\mathrm{sa}})^\vee) \to 0.$$
Taking the τ-component gives
$$0 \to \underline{\omega}_\tau \to \mathcal{H}^1_\tau \to \wedge^2(\mathcal{H}^1_\tau)\otimes\underline{\omega}_\tau^{-1} \to 0. \tag{2.11.1}$$
The line bundle $\wedge^2(\mathcal{H}^1_\tau)$ can be trivialized over $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$ using the prime-to-p polarization, but we will not need this fact.
3. There exists an $\mathcal{O}_F$-equivariant integral connection with logarithmic poles
$$\nabla : \mathcal{H}^1 \to \mathcal{H}^1 \otimes_{\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}} \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}),$$
which extends the Gauss-Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathbf{Sh}_{K,R})$.
4. Let KS be the map
$$\mathrm{KS} : \underline{\omega} \hookrightarrow \mathcal{H}^1 \xrightarrow{\nabla} \mathcal{H}^1 \otimes_{\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}} \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}) \to \mathrm{Lie}((\mathcal{A}^{\mathrm{sa}})^\vee)\otimes_{\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}} \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}).$$
It induces an *extended Kodaira-Spencer isomorphism* [**La13a**, 6.4.1.1]
$$\mathrm{Kod} : \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}(\log \mathsf{D}) \xrightarrow{\sim} \underline{\omega}\otimes_{(\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}\otimes\mathcal{O}_F)} \mathrm{Lie}(\mathcal{A}^\vee)^* \simeq \bigoplus_{\tau\in\Sigma_R}\underline{\Omega}_\tau, \tag{2.11.2}$$
where $\underline{\Omega}_\tau = \underline{\omega}_\tau^{\otimes 2}\otimes\wedge^2(\mathcal{H}^1_\tau)^{-1}$. For $J\subseteq\Sigma_R$, we put $\underline{\Omega}^J = \bigotimes_{\tau\in J}\underline{\Omega}_\tau$.

Let $\bigwedge^*_{\mathbb{Z}}(\mathbb{Z}[\Sigma_R])$ be the exterior algebra of the $\mathbb{Z}$-module $\mathbb{Z}[\Sigma_R]$, and $(e_\tau)_{\tau\in\Sigma_R}$ denote the natural basis. We fix an order on $\Sigma_R = \{\tau_1,\dots,\tau_g\}$. We put $e_\emptyset = 1$ and $e_J = e_{i_1}\wedge\cdots\wedge e_{i_j}$, for any subset $J = \{\tau_{i_1},\dots,\tau_{i_j}\}$ with $i_1 < \cdots < i_j$. We call these

e_J *Čech symbols*. Using them, we can write more canonically $\mathrm{Kod}:\Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}(\log \mathsf{D})\simeq \bigoplus_{\tau\in\Sigma_R}\underline{\Omega}_\tau e_\tau$. It induces an isomorphism of graded algebras

$$\Omega^\bullet_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D})=\bigoplus_{0\le j\le g}\Omega^j_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D})\simeq\bigoplus_{J\subseteq\Sigma_R}\underline{\Omega}^J e_J. \tag{2.11.3}$$

2.12. Integral models of automorphic bundles. — For a multiweight $(\underline{k},w)$, we put

$$\mathscr{F}^{(\underline{k},w)}_\tau:=(\wedge^2\mathcal{H}^1_\tau)^{\frac{w-k_\tau}{2}}\otimes \mathrm{Sym}^{k_\tau-2}\mathcal{H}^1_\tau,\quad\text{and}\quad \mathscr{F}^{(\underline{k},w)}:=\bigotimes_{\tau\in\Sigma_R}\mathscr{F}^{(\underline{k},w)}_\tau.$$

The extended Gauss-Manin connection on $\mathcal{H}^1$ induces by functoriality an integrable connection

$$\nabla:\mathscr{F}^{(\underline{k},w)}\to\mathscr{F}^{(\underline{k},w)}\otimes\Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}(\log \mathsf{D}).$$

By considering the associated local system $(\mathscr{F}^{(\underline{k},w)})^{\nabla=0}$ on $\mathbf{Sh}_K(\mathbb{C})$, it is easy to see that $(\mathscr{F}^{(\underline{k},w)},\nabla)$ on $\mathbf{Sh}^{\mathrm{tor}}_{K,\mathcal{O}_{F',(p)}}$ gives an integral model of the corresponding automorphic bundle on $\mathbf{Sh}_K(\mathbb{C})$ considered in Subsection 2.2. Similarly, we define a line bundle on $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$

$$\underline{\omega}^{(\underline{k},w)}:=\bigotimes_{\tau\in\Sigma_R}\Big((\wedge^2\mathcal{H}^1_\tau)^{\frac{w-k_\tau}{2}}\otimes\underline{\omega}_\tau^{k_\tau-2}\Big),$$

which is an integral model of the automorphic bundle (2.2.1). WE define *the space of Hilbert modular forms of weight* $(\underline{k},w)$ *and level* K *with coefficients in* R to be

$$M_{(\underline{k},w)}(K,R):=H^0(\mathbf{Sh}^{\mathrm{tor}}_{K,R},\underline{\omega}^{(\underline{k},w)}\otimes\Omega^g_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}(\log \mathsf{D})),$$

and the subspace of cusp forms to be

$$S_{(\underline{k},w)}(K,R):=H^0(\mathbf{Sh}^{\mathrm{tor}}_{K,R},\underline{\omega}^{(\underline{k},w)}\otimes\Omega^g_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}).$$

By Koecher's principle, one has $M_{(\underline{k},w)}(K,R)=H^0(\mathbf{Sh}_{K,R},\underline{\omega}^{(\underline{k},w)}\otimes\Omega^g_{\mathbf{Sh}_{K,R}})$, which coincides the definition (2.2.2) when R is a subfield of $\mathbb{C}$.

Suppose we are given $g\in G(\mathbb{A}^{\infty,p})$, and open subgroups $K'^p,K^p\subset G(\mathbb{A}^{\infty,p})$ with $g^{-1}K'^pg\subseteq K^p$. Let $K'=K'^pK'_p$ and $K=K^pK_p$ with $K'_p=K_p=\mathrm{GL}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$. Let $[g]^{\mathrm{tor}}:\mathbf{Sh}^{\mathrm{tor}}_{K',R}\to\mathbf{Sh}^{\mathrm{tor}}_{K,R}$ denote the morphism (2.10.1). Then according to [**La13a**, Theorem 2.15(4)], we have canonical isomorphisms of vector bundles on $\mathbf{Sh}^{\mathrm{tor}}_{K',R}$

$$[g]^{\mathrm{tor},*}(\mathscr{F}^{(\underline{k},w)})\xrightarrow{\simeq}\mathscr{F}^{(\underline{k},w)}, \tag{2.12.1}$$

compatible with the connection ∇ on $\mathscr{F}^{(\underline{k},w)}$ and the Hodge filtration to be defined in Subsection 2.14. Similarly, we have an isomorphism on $\mathbf{Sh}^{\mathrm{tor}}_{K',R}$: $[g]^{\mathrm{tor},*}(\underline{\omega}^{(\underline{k},w)})\xrightarrow{\simeq}\underline{\omega}^{(\underline{k},w)}$.

Remark 2.13. — Intuitively, the bundle $\mathcal{H}^1_\tau$ on $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$ "should be" the automorphic vector bundle corresponding to the representation $\check{\mathrm{St}}_\tau$ of $G_{\mathbb{C}}=(\mathrm{GL}_{2,\mathbb{C}})^{\Sigma_\infty}$ in the sense of [**Mil90a**, Chap. III]. But actually the representation $\check{\mathrm{St}}_\tau$ does not give rise to an automorphic vector bundle, because it does not factor through the quotient group $G^c_{\mathbb{C}}$ as explained in *loc. cit.*. Similarly, a line bundle of the form

$$\bigotimes_{\tau\in\Sigma_\infty}\big((\wedge^2\mathcal{H}^1_\tau)^{m_\tau}\otimes\underline{\omega}_\tau^{k_\tau}\big)$$

with $m_\tau,k_\tau\in\mathbb{Z}$, is an automorphic vector bundle in the sense of *loc. cit.* if and only if $2m_\tau+k_\tau=w$ is an integer independent of τ.

2.14. De Rham complex and Hodge filtrations. — We denote by $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ the de Rham complex

$$\mathscr{F}^{(\underline{k},w)} \xrightarrow{\nabla} \mathscr{F}^{(\underline{k},w)} \otimes \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathscr{F}^{(\underline{k},w)} \otimes \Omega^g_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}).$$

For a coherent sheaf $\mathscr{L}$ on $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$, we denote by $\mathscr{L}(-\mathsf{D})$ the tensor product of L with the ideal sheaf of D. For $0 \leq i \leq g-1$, ∇ induces a map

$$\nabla : \mathscr{F}^{(\underline{k},w)}(-\mathsf{D}) \otimes \Omega^i_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}) \to \mathscr{F}^{(\underline{k},w)}(-\mathsf{D}) \otimes \Omega^{i+1}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}).$$

We denote by $\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ the resulting complex by tensoring $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ with $\mathscr{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}(-\mathsf{D})$.

The complex $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ (and similarly for $\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$) is equipped with a natural Hodge filtration. Let (ω_τ, η_τ) be a local basis of $\mathcal{H}^1_\tau$ adapted to the Hodge filtration (2.11.1). We define $\mathtt{F}^n\mathscr{F}^{(\underline{k},w)}_\tau$ to be the submodule generated by the vectors

$$\{(\omega_\tau \wedge \eta_\tau)^{\frac{w-k_\tau}{2}} \otimes \omega^i_\tau \otimes \eta^{k_\tau-2-i}_\tau : n - \frac{w-k_\tau}{2} \leq i \leq k_\tau - 2\}$$

if $\frac{w-k_\tau}{2} \leq n \leq \frac{w-k_\tau}{2} + k_\tau - 2$, and

$$\mathtt{F}^n\mathscr{F}^{(\underline{k},w)}_\tau = \begin{cases} \mathscr{F}^{(\underline{k},w)}_\tau & \text{if } n \leq \frac{w-k_\tau}{2} \\ 0 & \text{if } n \geq \frac{w-k_\tau}{2} + k_\tau - 1. \end{cases} \tag{2.14.1}$$

The filtration does not depend on the choice of (ω_τ, η_τ), and the graded pieces of the filtration are

$$\mathrm{Gr}^n_{\mathtt{F}}\mathscr{F}^{(\underline{k},w)}_\tau \simeq \begin{cases} (\wedge^2\mathcal{H}^1_\tau)^{w-n-2} \otimes \underline{\omega}^{2n+2-w}_\tau & \text{if } n \in [\frac{w-k_\tau}{2}, \frac{w+k_\tau}{2} - 2] \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the sheaf $\mathscr{F}^{(\underline{k},w)}$. We endow it with the tensor product filtration induced by $(\mathtt{F}^n\mathscr{F}^{(\underline{k},w)}_\tau, n \in \mathbb{Z})$ for $\tau \in \Sigma_R$. The $\mathtt{F}$-filtration on $\mathscr{F}^{(\underline{k},w)}$ satisfies Griffiths' traversality for ∇, i.e. we have

$$\nabla : \mathtt{F}^n\mathscr{F}^{(\underline{k},w)} \to \mathtt{F}^{n-1}\mathscr{F}^{(\underline{k},w)} \otimes \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}/R}(\log \mathsf{D}).$$

We define $\mathtt{F}^n\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ as the subcomplex $\mathtt{F}^{n-\bullet}\mathscr{F}^{(\underline{k},w)} \otimes \Omega^\bullet_{\mathrm{Sh}^{\mathrm{tor}}_R/R}(\log \mathsf{D})$ of $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$, and call it the $\mathtt{F}$-filtration (or Hodge filtration) on $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$. The $\mathtt{F}$-filtration on $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ induces naturally an $\mathtt{F}$-filtration on $\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$.

2.15. The dual BGG-complex. — Assume that $(k_\tau - 2)!$ is invertible in R for every $\tau \in \Sigma_R$. It is well known that $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ (resp. $\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$) is quasi-isomorphic to a much simpler complex $\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)})$ (resp. $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$), called the dual BGG-complexes of $\mathscr{F}^{(\underline{k},w)}$. Here, we tailor the discussion for later application and refer the reader to [**Fa82**, §3 and §7] and [**LP11**] for details.

The Weyl group of $G_R = (\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2)_R$ is canonically isomorphic to $W_G = \{\pm 1\}^{\Sigma_R}$. For a subset $J \subseteq \Sigma_R$, let $\epsilon_J \in W_G = \{\pm 1\}^{\Sigma_R}$ be the element whose τ-component is -1 for $\tau \notin J$ and is 1 for $\tau \in J$. In particular, ϵ_{Σ_R} is the identity element of W_G. We define

$$\underline{\omega}^{\epsilon_J(\underline{k},w)} :\, = \Big(\big(\bigotimes_{\tau\notin J} (\wedge^2\mathcal{H}^1_\tau)^{\frac{w+k_\tau}{2}-2} \otimes \underline{\omega}^{2-k_\tau}_\tau\big) \otimes \big(\bigotimes_{\tau\in J} (\wedge^2\mathcal{H}^1_\tau)^{\frac{w-k_\tau}{2}} \otimes \underline{\omega}^{k_\tau-2}_\tau\big)\Big). \tag{2.15.1}$$

This is an integral model of an automorophic vector bundle (See Remark 2.13). For any $0 \leq j \leq g$, we put

$$\mathrm{BGG}^j(\mathscr{F}^{(\underline{k},w)}) = \bigoplus_{\substack{J\subseteq\Sigma_R \\ \#J=j}} \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J e_J. \tag{2.15.2}$$

There exists a differential operator $d^j : \mathrm{BGG}^j(\mathscr{F}^{(\underline{k},w)}) \to \mathrm{BGG}^{j+1}(\mathscr{F}^{(\underline{k},w)})$ described as follows: for a local section f of $\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J \subseteq \mathrm{BGG}^j(\mathscr{F}^{(\underline{k},w)})$ with $\#J = j$, we define

$$d^j \colon fe_J \mapsto \sum_{\tau_0 \notin J} \Theta_{\tau_0, k_{\tau_0}-1}(f) e_{\tau_0} \wedge e_J. \tag{2.15.3}$$

Here, $\Theta_{\tau_0,k_{\tau_0}-1}$ is a certain differential operator of order $k_{\tau_0} - 1$ (See Remark 2.17(1)), and it is an analog of the classical theta operator. We define a decreasing $\mathtt{F}$-filtration on $\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)})$ by setting:

$$\mathtt{F}^n\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)}) = \bigoplus_{\substack{J \subseteq \Sigma_R \\ n_J \geq n}} \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J \; e_J[-\#J]$$

where $n_J \colon = \sum_{\tau \in J}(k_\tau - 1) + \sum_{\tau \in \Sigma_R} \frac{w-k_\tau}{2}$. It is easy to see that $\mathtt{F}^n\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)})$ is stable under the differentials d^j, and the graded pieces

$$\mathrm{Gr}^n_{\mathtt{F}}\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)}) = \bigoplus_{\substack{J \subseteq \Sigma_R \\ n_J = n}} \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J \; e_J[-\#J] \tag{2.15.4}$$

have trivial induced differentials.

Finally, the differential d^j preserves cuspidality, i.e. it induces a map

$$d^j : \mathrm{BGG}^j(\mathscr{F}^{(\underline{k},w)})(-\mathsf{D}) \to \mathrm{BGG}^{j+1}(\mathscr{F}^{(\underline{k},w)})(-\mathsf{D}).$$

we will denote by $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ the resulting complex. The $\mathtt{F}$-filtration on $\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)})$ induces an $\mathtt{F}$-filtration on $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$, and the graded pieces $\mathrm{Gr}^n_{\mathtt{F}}(\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)}))$ are given by (2.15.4) twisted by $\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}(-\mathsf{D})$.

Theorem 2.16 **(Faltings; cf. [Fa82] §3 and §7, [FC90]Chap. §5, [LP11] §5)**
Assume that $(k_\tau - 2)!$ is invertible in R for each $\tau \in \Sigma_R$. Then there is a canonical quasi-isomorphic embedding of $\mathtt{F}$-filtered complexes of abelian sheaves on $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$

$$\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)}) \hookrightarrow \mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)}).$$

Similarly, we have a canonical quasi-isomorphism of $\mathtt{F}$-filtered complexes

$$\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)}) \hookrightarrow \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)}).$$

Remark 2.17. — (1) It is possible to give an explicit formula for the operator $\Theta_{\tau_0,k_{\tau_0}-1}$ appearing in (2.15.3). Let f be a local section of $\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J$ with q-expansion

$$f = \sum_\xi a_\xi q^\xi$$

at a cusp of $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$, where ξ runs through 0 and the set of totally positive elements in a lattice of F. Using the complex uniformization, one can show that

$$\Theta_{\tau_0,k_{\tau_0}-1}(f) = \frac{(-1)^{k_{\tau_0}-2}}{(k_{\tau_0}-2)!} \sum_\xi \tau_0(\xi)^{k_{\tau_0}-1} a_\xi q^\xi.$$

The denominator $(k_{\tau_0} - 2)!$ explains the assumption that $(k_\tau - 2)!$ is invertible in R for every τ. The main results of this paper do not use this formula on q-expansions.

(2) The embedding $\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)}) \hookrightarrow \mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ is constructed using reprensentation theory, and the morphisms in each degree are given by differential operators rather than morphisms of $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_R}$-modules (cf. [**Fa82**, §3] and [**FC90**, Chap. VI §5]). When $k_\tau = 3$ for all $\tau \in \Sigma_R$, the embedding $\mathrm{BGG}^\bullet(\mathscr{F}^{(\underline{k},w)}) \hookrightarrow \mathscr{F}^{(\underline{k},w)}$ splits the Hodge filtration on $\mathcal{H}^1$ globally as abelian sheaves over $\mathbf{Sh}^{\mathrm{tor}}_{K,R}$, and it is certainly not $\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}}$-linear.

(3) Assume $R=\mathbb{C}$. Let $\mathbb{L}^{(\underline{k},w)}$ denote the local system $\mathscr{F}^{(\underline{k},w)}(\mathbb{C})^{\nabla=0}$ on the complex manifold $\mathbf{Sh}_K(\mathbb{C})$, and $j:\mathbf{Sh}_K(\mathbb{C})\hookrightarrow\mathbf{Sh}_K^{\mathrm{tor}}(\mathbb{C})$ be the open immersion. Then by the Riemann-Hilbert-Deligne correspondence, $\mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})$ resolves $Rj_*(\mathbb{L}^{(\underline{k},w)})$, and $\mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$ resolves the sheaf $j_!(\mathbb{L}^{(\underline{k},w)})$ [**FC90**, Chap. VI. 5.4].

3. Overconvergent Hilbert Modular Forms

3.1. Notation. — We fix a number field $L\subset\mathbb{C}$ containing $\tau(F)$ for all $\tau\in\Sigma_\infty$. The fixed isomorphism $\iota_p:\mathbb{C}\xrightarrow{\simeq}\overline{\mathbb{Q}}_p$ determines a p-adic place $\wp$ of L. We denote by $L_\wp$ the completion, $\mathcal{O}_\wp$ the ring of integers, and k_0 the residue field. The isomorphism ι_p also identifies Σ_∞ with the set of p-adic embeddings $\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)=\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F,k_0)$. The natural action of the Frobenius on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F,k_0)$ defines, via the identification above, a natural action on Σ_∞: $\tau\mapsto\sigma\circ\tau$. We have a natural partition: $\Sigma_\infty=\coprod_{\mathfrak{p}\in\Sigma_p}\Sigma_{\infty/\mathfrak{p}}$, where $\Sigma_{\infty/\mathfrak{p}}$ consists of the τ's such that $\iota_p\circ\tau$ induces the place $\mathfrak{p}$. For any $\mathcal{O}_\wp$-scheme S and a coherent $(\mathcal{O}_S\otimes\mathcal{O}_F)$-sheaf M, we have a canonical decomposition $M=\bigoplus_{\tau\in\Sigma_\infty}M_\tau$, where M_τ is the direct summand of M on which $\mathcal{O}_F$ acts via $\tau:\mathcal{O}_F\to\mathcal{O}_\wp\to\mathcal{O}_S$.

Unless stated otherwise, we take the open compact subgroups $K=K^pK_p\subset G(\mathbb{A}^\infty)$ so that $K_p=\mathrm{GL}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$ and that K^p satisfies Hypothesis 2.7; then the corresponding Shimura variety $\mathbf{Sh}_K(G)$ is a fine moduli space of abelian varieties over $\mathbb{Z}_{(p)}$. We choose a toroidal compactification $\mathbf{Sh}_K^{\mathrm{tor}}(G)$, and let $\mathbf{Sh}_K^*(G)$ be the minimal compactification as in Subsection 2.10. To simplify notation, let $\mathbf{X}_K$, $\mathbf{X}_K^{\mathrm{tor}}$, and $\mathbf{X}_K^*$ denote the base change to $W(k_0)$ of $\mathbf{Sh}_K(G)$, $\mathbf{Sh}_K^{\mathrm{tor}}(G)$, and $\mathbf{Sh}_K^*(G)$, respectivley. Let X_K, X_K^{tor} and X_K^* be respectively their special fibers. Denote by $\mathfrak{X}_K^{\mathrm{tor}}$ the formal completion of $\mathbf{X}_K^{\mathrm{tor}}$ along X_K^{tor}, and $\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}$ the base change to $L_\wp$ of the associated rigid analytic space over $W(k_0)[1/p]$. For a locally closed subset $U_0\subset X_K$, let $]U_0[$ be the inverse image of U_0 under the specialization map $\mathrm{sp}:\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}\to X_K^{\mathrm{tor}}$. Similarly, we have the evident variants $\mathfrak{X}_K^*$, $\mathfrak{X}_{K,\mathrm{rig}}^*$ for the minimal compactification $\mathbf{X}^*$. If there is no risk of confusion, we will use the same notation D to denote the toroidal boundary in various settings: $\mathbf{X}_K^{\mathrm{tor}}-\mathbf{X}_K$ and $X_K^{\mathrm{tor}}-X_K$.

3.2. Hasse invariant and ordinary locus. — Let $\mathcal{A}_{k_0}^{\mathrm{sa}}$ be the semi-abelian scheme over X_K^{tor} that extends the universal abelian variety $\mathcal{A}_{k_0}$ over X_K. The Verschiebung homomorphism $\mathrm{Ver}:(\mathcal{A}_{k_0}^{\mathrm{sa}})^{(p)}\to\mathcal{A}_{k_0}^{\mathrm{sa}}$ induces an $\mathcal{O}_F$-linear map on the module of invariant differential 1-forms:

$$h:\underline{\omega}\to\underline{\omega}^{(p)},$$

which induces, for each $\tau\in\Sigma_\infty$ (identified with the set of p-adic embeddings of F), a map $h_\tau:\underline{\omega}_\tau\to\underline{\omega}^p_{\sigma^{-1}\circ\tau}$. This defines, for each $\tau\in\Sigma_\infty$, a section

$$h_\tau\in H^0(X_K^{\mathrm{tor}},\underline{\omega}^p_{\sigma^{-1}\circ\tau}\otimes\underline{\omega}_\tau^{-1}).$$

We put $h=\otimes_{\tau\in\Sigma_\infty}h_\tau\in\Gamma(X_K^{\mathrm{tor}},\det(\underline{\omega})^{p-1})$. We call h and h_τ respectively the *(total) Hasse invariant*, and the *partial Hasse invariant at* τ.

Let Y_K and $Y_{K,\tau}$ be the closed subschemes of X_K^{tor} defined by the vanishing locus of h and h_τ. Each $Y_{K,\tau}$ is reduced and smooth, and $Y_K=\bigcup_\tau Y_{K,\tau}$ is a normal crossing divisor in X_K^{tor} [**GO00**]. We call the complement $X_K^{\mathrm{tor,ord}}=X_K^{\mathrm{tor}}-Y_K$ the *ordinary locus*. This is the open subscheme of the moduli space X_K^{tor} where the semi-abelien scheme $\mathcal{A}_{k_0}^{\mathrm{sa}}$ is ordinary. We point out that Y_K does not intersect the toroidal boundary $\mathsf{D}=X_K^{\mathrm{tor}}-X_K$.

Similarly, for the minimal compactification X_K^*, we put $X_K^{*,\mathrm{ord}}=X_K^*-Y_K$. Since $\det(\underline{\omega})$ is an ample line bundle on X_K^* (See Subsection 2.10), $X_K^{*,\mathrm{ord}}$ is affine.

3.3. Overconvergent Cusp Forms. — Let $j:]X_K^{\mathrm{tor,ord}}[\hookrightarrow\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}$ be the natural inclusion of rigid analytic spaces. When it is necessary, we write j_K instead to emphasize the level K.

For a coherent sheaf $\mathcal{F}$ on $\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$, following Berthelot [**Be96**], we define $j^\dagger\mathcal{F}$ to be the sheaf on $\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$ such that, for all admissible open subset $U \subset X^{\mathrm{tor}}_{K,\mathrm{rig}}$, we have

$$\Gamma(U, j^\dagger\mathcal{F}) = \varinjlim_{V} \Gamma(V \cap U, \mathcal{F}),$$

where V runs through a fundamental system of strict neighborhoods of $]X^{\mathrm{tor,ord}}_K[$ in $\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$. An explicit fundamental system of strict neighborhoods of $]X^{\mathrm{tor,ord}}_K[$ in $\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$ can be constructed as follows. Let $\tilde{E}$ be a lift to characteristic 0 of a certain power of the Hasse invariant h. For any rational number $r > 0$, we denote by $]X^{\mathrm{tor,ord}}_K[_r$ the admissible open subset of $\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$ defined by $|\tilde{E}| \le p^{-r}$. Then the admissible open subsets $]X^{\mathrm{tor,ord}}_K[_r$ with $r \to 0^+$ form a fundamental system of strict neighborhoods of $]X^{\mathrm{tor,ord}}_K[$ in $\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$. For the minimal compactification $\mathbf{X}^*_K$, we can define similarly admissible open subsets $]X^{*,\mathrm{ord}}_K[_r$, which also form a fundamental system of strict neighborhoods of $]X^{*,\mathrm{ord}}_K[$ in $\mathfrak{X}^*_{\mathrm{rig}}$. We again point out that $]X^{*,\mathrm{ord}}_K[_r$ are affinoid subdomains of $\mathfrak{X}^*_{K,\mathrm{rig}}$, while $]X^{\mathrm{tor,ord}}_K[_r$ are not.

For a multiweight $(\underline{k}, w)$, we define the space of *cuspidal overconvergent Hilbert modular forms* (*overconvergent cusp forms* for short) with coefficients in $L_\wp$ to be

$$S^\dagger_{(\underline{k},w)}(K, L_\wp) :\, = H^0(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger\underline{\omega}^{(\underline{k},w)} \otimes \Omega^g_{\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}/L_\wp}).$$

When there is no risk of confusions, we write $S^\dagger_{(\underline{k},w)}$ for $S^\dagger_{(\underline{k},w)}(K, L_\wp)$.

The space of overconvergent cusp forms contains all classical cusp forms with Iwahoric level structure at p: we denote by $\mathrm{Iw}_p = \prod_{\mathfrak{p}|p} \mathrm{Iw}_{\mathfrak{p}} \subset \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ the Iwahoric subgroup, where

$$\mathrm{Iw}_{\mathfrak{p}} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}}) \;\middle|\; c \equiv 0 \mod \mathfrak{p} \right\}. \tag{3.3.1}$$

Let $S_{(\underline{k},w)}(K^p\mathrm{Iw}_p, L_\wp)$ be the space of classical cusp forms of multiweight $(\underline{k}, w)$ and of prime-to-p level K^p and Iwahoric level at all places above p. By the theory of canonical subgroups, there is a natural injection

$$\iota : S_{(\underline{k},w)}(K^p\mathrm{Iw}_p, L_\wp) \hookrightarrow S^\dagger_{(\underline{k},w)}(K, L_\wp).$$

An overconvergent cusp form $f \in S^\dagger_{(\underline{k},w)}(K, L_\wp)$ is called *classical*, if it lies in the image of ι.

For each subset $J \subseteq \Sigma_\infty$, recall that $\epsilon_J \in \{\pm1\}^{\Sigma_\infty}$ is the element which is 1 at $\tau \in J$ and -1 at $\tau \notin J$. We put

$$S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp) = \varinjlim_{U} H^0(U, \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})), \tag{3.3.2}$$

where $\underline{\omega}^{\epsilon_J(\underline{k},w)}$ is defined in (2.11.3) and $\underline{\Omega}^J$ is defined just below (2.11.3). As usual, when the context is clear, we write $S^\dagger_{\epsilon_J(\underline{k},w)} = S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp)$. In particular, when $J = \Sigma_\infty$ we have $S^\dagger_{\epsilon_{\Sigma_\infty}\cdot(\underline{k},w)} = S^\dagger_{(\underline{k},w)}$.

We remark that, by Kodaira-Spencer isomorphism (2.11.2), we can identify $S^\dagger_{\epsilon_J(\underline{k},w)}$ with $S^\dagger_{(\underline{k}',w)}$, where $k'_\tau = k_\tau$ if $\tau \in J$ and $k'_\tau = 2 - k_\tau$ if $\tau \notin J$. But we prefer to use (3.3.2) because keeping the differential forms reminds us the sheaf is part of the dual BGG complex.

Recall that the dual BGG-complex $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ is quasi-isomorphic to the de Rham complex $\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ (Theorem 2.16). By applying $j^\dagger$ to $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ and taking global sections, we get a complex $\mathscr{C}^\bullet_K$ of overconvergent cusp forms concentrated in degrees $[0, g]$ with

$$\mathscr{C}^j_K :\, = \bigoplus_{\substack{J \subseteq \Sigma_\infty \\ \#J = j}} S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp) e_J. \tag{3.3.3}$$

Here, e_J is the symbol introduced in (2.11.3) in order to get the correct signs, and the differential map $d^j : \mathscr{C}_K^j \to \mathscr{C}_K^{j+1}$ is given by (2.15.3).

3.4. Rigid cohomology of the ordinary locus.— We denote by $j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$ the complex of sheaves on $\mathfrak{X}_{\mathrm{rig}}^{\mathrm{tor}}$ by applying $j^\dagger$ to each component of $\mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$. We define

$$R\Gamma_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)}) := R\Gamma(\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)}))$$

as an object in the derived category of L-vector spaces, and its cohomology groups will be denoted by

$$H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)}) := \mathbb{H}^\star(\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})),$$

where the left hand side denotes the hypercohomology of the complex $j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$. In Section 4, we will interpret $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ as the rigid cohomology of a certain isocrystal over the ordinary locus $X_K^{\mathrm{tor,ord}}$ and with compact support in $\mathsf{D} \subset X_K^{\mathrm{tor}}$.

Theorem 3.5. — *The object $R\Gamma_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ in the derived category of L-vector spaces is represented by the complex $\mathscr{C}_K^\bullet$ defined in* (3.3.3). *In particular, we have an isomorphism*

$$H^g_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)}) \cong S^\dagger_{(\underline{k},w)} / \sum_{\tau \in \Sigma_\infty} \Theta_{\tau, k_\tau - 1}(S^\dagger_{\epsilon_{\Sigma_\infty \setminus \{\tau\}} \cdot (\underline{k},w)}).$$

The following Lemma is due to Kai-Wen Lan.

Lemma 3.6 (**[La13b] Theorem 8.2.1.3**). — *Let $\pi : \mathbf{X}_{K,L}^{\mathrm{tor}} \to \mathbf{X}_{K,L}^*$ be the natural projection. Then for any subset $J \subseteq \Sigma_\infty$, we have $R^q \pi_*(\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})) = 0$ for $q > 0$.*

Proof of Theorem 3.5. — Since the complex $\mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$ is quasi-isomorphic to the compact supported de Rham complex $\mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$, we have

$$R\Gamma_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)}) \cong R\Gamma(\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger \mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) \cong R\Gamma(\mathfrak{X}_{\mathrm{rig}}^*, R\pi_* j^\dagger \mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)})).$$

Since the boundary $\mathsf{D} \subset \mathfrak{X}_{\mathrm{rig}}^{\mathrm{tor}}$ is contained in the ordinary locus $]X_K^{\mathrm{tor,ord}}[$, we have $R\pi_* j^\dagger = j^\dagger R\pi_*$. By Lemma 3.6, we have $R\pi_* \mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)}) = \pi_* \mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$. Let $]X_K^{*,\mathrm{ord}}[_r$ for rational $r > 0$ be the strict neighborhoods of $]X_K^{*,\mathrm{ord}}[$ considered in Subsection 3.3. Since the $]X_K^{*,\mathrm{ord}}[_r$'s are affinoid and form a fundamental system of strict neighborhoods of $]X_K^{*,\mathrm{ord}}[$ in $\mathfrak{X}_{\mathrm{rig}}^*$, we deduce that

$$\begin{aligned} &H^n(\mathfrak{X}_{K,\mathrm{rig}}^*, j^\dagger \pi_* \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})) \\ &= \varinjlim_{r \to 0} H^n(]X_K^{*,\mathrm{ord}}[_r, \pi_* \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})) = \begin{cases} 0 & \text{for } n \neq 0, \\ S^\dagger_{\epsilon_J(\underline{k},w)} & \text{for } n = 0. \end{cases} \end{aligned}$$

It follows that

$$R\Gamma(\mathfrak{X}_{K,\mathrm{rig}}^*, j^\dagger \pi_* \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})) = S^\dagger_{\epsilon_J(\underline{k},w)},$$

and hence

$$R\Gamma_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) = R\Gamma(\mathfrak{X}_{K,\mathrm{rig}}^*, j^\dagger \pi_* \mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) = \mathscr{C}_K^\bullet.$$

This finishes the proof of the theorem. □

3.7. Prime-to-p Hecke actions. — Let $\mathscr{H}(K^p, L_\wp) = L_\wp[K^p\backslash G(\mathbb{A}^{\infty,p})/K^p]$ be the prime-to-p Hecke algebra of level K^p. We will define actions of $\mathscr{H}(K^p, L_\wp)$ on $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ and on the complex $\mathscr{C}^\bullet_K$ such that the actions are compatible via Theorem 3.5.

Consider the double coset $[K^pgK^p]$. We put $K'^p = K^p \cap gK^pg^{-1}$ and $K' = K'^pK_p$. By choosing suitable rational polyhedral cone decomposition data, we have the following Hecke correspondence (2.10.2):

$$\begin{array}{ccc} & \mathbf{X}^{\mathrm{tor}}_{K'} & \\ {}^{\pi_1=[1]^{\mathrm{tor}}}\swarrow & & \searrow^{\pi_2=[g]^{\mathrm{tor}}} \\ \mathbf{X}^{\mathrm{tor}}_{K'} & & \mathbf{X}^{\mathrm{tor}}_{K}. \end{array}$$

In view of the isomorphism (2.11.3), one has a natural map of complexes of sheaves

$$\pi_2^*\colon \pi_2^{-1}\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)}) \to \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)}),$$

which is compatible with the F-filtration. For each $J \subseteq \Sigma_\infty$, the sheaf $\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}_K)$ appears as a direct summand of $\mathrm{Gr}^\bullet_{\mathsf{F}}\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ by Theorem 2.16. The morphism above induces a map of abelian sheaves:

$$\pi_2^*\colon \pi_2^{-1}(\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}_K)) \to \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}_{K'}).$$

Here, to avoid confusions, we use subscripts to distinguish the toroidal boundaries D_K of $\mathbf{X}^{\mathrm{tor}}_K$ and $\mathsf{D}_{K'}$ of $\mathbf{X}^{\mathrm{tor}}_{K'}$ respectively. It is clear that, by construction, the resulting morphism of BGG-complexes

$$\pi_2^* : \pi_2^{-1}\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)}) \to \mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$$

is compatible with the natural quasi-isomorphic inclusion $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)}) \hookrightarrow \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$.

Lemma 3.8. — *Under the above notation, we have $R^q\pi_{1,*}\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_{K'}} = 0$ for $q > 0$, and $\pi_{1,*}(\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_{K'}})$ is finite flat over $\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_{K}}$.*

Proof. — The statement is clear over $\mathbf{X}_K$, since π_1 is finite étale there. Therefore, it is enough to prove the lemma after base change π_1 to the completion of $\mathbf{X}^{\mathrm{tor}}_K$ along $\mathsf{D}_K = \mathbf{X}^{\mathrm{tor}}_K - \mathbf{X}_K$. Then the morphism π_1 over the completion is étale locally given by equivariant morphisms between toric varieties, the results follow from similar arguments as in [**KKMS**, Ch. I §3]. □

Corollary 3.9. — *There exist natural trace maps $\mathrm{Tr}_{\pi_1} : R\pi_{1,*}\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)}) \to \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ and*

$$\mathrm{Tr}_{\pi_1}\colon R\pi_{1,*}\big(\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}_{K'})\big) \to \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}_K)$$

for each $J \subseteq \Sigma_\infty$, such that the induced map Tr_{π_1} on $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ is compatible with that on $\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$ via the quasi-isomorphism of Theorem 2.16.

Proof. — By (2.12.1), each term $M' = \mathrm{DR}^j_c(\mathscr{F}^{(\underline{k},w)})$ or $\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}_{K'})$ on $\mathbf{X}^{\mathrm{tor}}_{K'}$ is the pullback via π_1 of the corresponding object on $\mathbf{X}^{\mathrm{tor}}_K$, i.e. M' has the form $M' = \pi_1^*(M)$. By the projection formula and Lemma above, we have $R\pi_{1,*}(M') \simeq M \otimes_{\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_K}} R\pi_{1,*}(\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_{K'}}) = M \otimes_{\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_K}} \pi_{1,*}\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_{K'}}$. The existence of the trace map $\pi_{1,*}(\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_{K'}}) \to \mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_K}$ follows from the finite flatness of $\pi_{1,*}(\mathcal{O}_{\mathbf{X}^{\mathrm{tor}}_{K'}})$. □

We can describe now the action of the double coset $[K^pgK^p]$ on $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ and $\mathscr{C}^\bullet_K$. Since the partial Hasse invariants depend only on the p-divisible group associated with the universal abelian scheme, it is clear that the inverse image of $X_K^{\mathrm{tor,ord}}$

by both π_1 and π_2 are identified with $X_{K'}^{\mathrm{tor,ord}}$. We define the action of $[K^pgK^p]$ on $R\Gamma(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j_K^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)}))$ to be the composite map:

$$\begin{array}{ccc} R\Gamma(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j_K^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) & \xrightarrow{\pi_2^*} & R\Gamma(\mathfrak{X}^{\mathrm{tor}}_{K',\mathrm{rig}}, j_{K'}^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) \\ & \searrow^{[K^pgK^p]_*} & \downarrow \mathrm{Tr}_{\pi_1} \\ & & R\Gamma(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j_K^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})), \end{array}$$

where Tr_{π_1} is induced by the trace map $\mathrm{Tr}_{\pi_1}: R\pi_{1,*}(\mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) \to \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$. Taking cohomology, one gets the actions of $[K^pgK^p]$ on the cohomology groups $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$, hence the action of $\mathscr{H}(K^p, L_\wp)$ by linear combinations.

Similarly, for each $J \subseteq \Sigma_\infty$, we define the action of $[K^pgK^p]$ on $S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp) = H^0(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j_K^\dagger \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}))$ to be

$$[K^pgK^p]_*: S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp) \xrightarrow{\pi_2^*} S^\dagger_{\epsilon_J(\underline{k},w)}(K', L_\wp) \xrightarrow{\mathrm{Tr}_{\pi_1}} S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp).$$

Putting together all $S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp)$, one gets the action of $[K^pgK^p]$ on the complex $\mathscr{C}_K^\bullet$. By construction, this action is compatible with the action on $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$.

3.10. The operator $S_\mathfrak{p}$. — We now define the Hecke actions at p. We start with the operator $S_\mathfrak{p}$ for $\mathfrak{p} \in \Sigma_p$. We define $[\varpi_\mathfrak{p}] : \mathbf{X}_K^{\mathrm{tor}} \to \mathbf{X}_K^{\mathrm{tor}}$ to be the endomorphism whose effect at non-cusp points are given by

$$S_\mathfrak{p}: (A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p}) \mapsto (A \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1}, \iota', \bar{\lambda}', \bar{\alpha}'_{K^p}),$$

where the induced structures on $A \otimes \mathfrak{p}^{-1} \cong A/A[\mathfrak{p}]$ are given as follows: The action ι' by $\mathcal{O}_F$ on $A \otimes \mathfrak{p}^{-1}$ is evident. The polarization λ' is given by

$$\lambda' : \mathfrak{c}_1\mathfrak{p}^2 \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{Sym}}_{\mathcal{O}_F}(A, A^\vee) \otimes_{\mathcal{O}_F} \mathfrak{p}^2 = \mathrm{Hom}^{\mathrm{Sym}}_{\mathcal{O}_F}(A \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1}, (A \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1})^\vee).$$

Finally, the level-K^p structure α_{K^p} on A induces naturally a level K^p-structure α'_{K^p} on $A \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1}$, since A and $A \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1}$ have naturally isomorphic prime-to-p Tate modules. This well defines a point $(A \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1}, \iota', \bar{\lambda}', \bar{\alpha}'_{K^p})$ on $\mathbf{X}_K$ with the convention in Remark 2.8.

The automorphism $S_\mathfrak{p}$ preserves the ordinary locus $X_K^{\mathrm{tor,ord}}$, since A and $A \otimes \mathfrak{p}^{-1}$ have isomorphic p-divisible groups and hence the same partial Hasse invariants. We have a canonical isogeny

$$[\varpi_\mathfrak{p}]: \mathcal{A}^{\mathrm{sa}} \to S_\mathfrak{p}^*(\mathcal{A}^{\mathrm{sa}}) = \mathcal{A}^{\mathrm{sa}} \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1},$$

with kernel $\mathcal{A}^{\mathrm{sa}}[\mathfrak{p}]$. It induces a map on the relative de Rham cohomology $[\varpi_\mathfrak{p}]^*: S_\mathfrak{p}^*\mathcal{H}^1 \to \mathcal{H}^1$, hence a morphism of vector bundles

$$[\varpi_\mathfrak{p}]^*: S_\mathfrak{p}^*\mathscr{F}^{(\underline{k},w)} \to \mathscr{F}^{(\underline{k},w)}$$

compatible with all the structures. The morphism $[\varpi_\mathfrak{p}]^*$ on $\mathscr{F}^{(\underline{k},w)}$ descends to $\mathbf{X}_K^{\mathrm{tor}}$ and induces a map on the de Rham complex

$$[\varpi_\mathfrak{p}]^*: \mathrm{DR}_c^\bullet(S_\mathfrak{p}^*\mathscr{F}^{(\underline{k},w)}) \to \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)}),$$

compatible with the F-filtrations. We define the action of $S_\mathfrak{p}$ on $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ to be the composite

(3.10.1)
$$\begin{array}{ccc} H^\star(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) & \xrightarrow{S_\mathfrak{p}^*} & H^\star(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger \mathrm{DR}_c^\bullet(S_\mathfrak{p}^*\mathscr{F}^{(\underline{k},w)})) \\ & \searrow^{S_\mathfrak{p}} & \downarrow [\varpi_\mathfrak{p}]^* \\ & & H^\star(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) \end{array}$$

Similarly, the morphism $[\varpi_{\mathfrak{p}}]^*$ on the de Rham complexes induces a map of the BGG-complexes

$$[\varpi_{\mathfrak{p}}]^* : \mathrm{BGG}_c^\bullet([\varpi_{\mathfrak{p}}]^*\mathscr{F}^{(\underline{k},w)}) \to \mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)}).$$

Using this, one defines an action of $S_{\mathfrak{p}}$ on each $S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp)$ for $J \subseteq \Sigma_\infty$, such that its resulting action on $\mathscr{C}_K^\bullet$ is compatible with that on $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$.

In the classical adelic language, the operator $S_{\mathfrak{p}}$ is the Hecke action given by $\left(\begin{smallmatrix}\varpi_{\mathfrak{p}}^{-1} & 0\\ 0 & \varpi_{\mathfrak{p}}^{-1}\end{smallmatrix}\right)$, where $\varpi_{\mathfrak{p}} \in \mathbb{A}_F^{\infty,\times}$ is a finite idèle which is a uniformizer of $F_{\mathfrak{p}}$ at $\mathfrak{p}$ and is 1 at other places.

3.11. The $\mathfrak{p}$-canonical subgroup. — For a rigid point $x \in \mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$ and $\tau \in \Sigma_\infty$, Goren and Kassaei defined in [**GK09**, 4.2] the *τ-valuation* of x, denoted by $\nu_\tau(x)$, as the truncated p-adic valuation of the τ-th partial Hasse invariant $h_\tau(\bar{x})$, where $\bar{x}$ denotes the reduction modulo p of x. Then $\nu_\tau(x)$ is a well defined rational number in $[0,1]$, and x belongs to the ordinary locus $]X_K^{\mathrm{tor,ord}}[$ if and only if $\nu_\tau(x) = 0$ for all $\tau \in \Sigma_\infty$. Let $\underline{r} = (r_{\mathfrak{q}}) \in [0,p)^{\Sigma_p}$ with $r_{\mathfrak{q}} \in \mathbb{Q}$. Following [**GK09**, 5.3], we put

$$]X_K^{\mathrm{tor,ord}}[_{\underline{r}} = \{x \in \mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}} \mid \nu_\tau(x) + p\nu_{\sigma^{-1}\circ\tau}(x) \le r_{\mathfrak{q}}\ \forall \tau \in \Sigma_{\infty/\mathfrak{q}}\}.$$

Then we have $]X_K^{\mathrm{tor,ord}}[_{\underline{r}} =]X_K^{\mathrm{tor,ord}}[$ for $\underline{r} = 0$, and $]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$ form a fundamental system of strict neighborhoods of $]X_K^{\mathrm{tor,ord}}[$ in $\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}$ as $r_{\mathfrak{q}} \to 0^+$ for all $\mathfrak{q} \in \Sigma_p$. We put $]X_K^{\mathrm{ord}}[_{\underline{r}} =]X_K[\cap]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$.

Now we fix a prime ideal $\mathfrak{p} \in \Sigma_p$, and choose $\underline{r} = (r_{\mathfrak{q}})_{\mathfrak{q}\in\Sigma_p}$ as above with $0 < r_{\mathfrak{p}} < 1$. Goren-Kassaei proved that there exists a finite flat subgroup scheme $\mathcal{C}_{\mathfrak{p}} \subset \mathcal{A}^{\mathrm{sa}}[\mathfrak{p}]$ over $]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$, called the universal $\mathfrak{p}$-canonical subgroup, satisfying the following properties [**GK09**, 5.3, 5.4]:

1. Locally for the étale topology on $]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$, we have $\mathcal{C}_{\mathfrak{p}} \simeq \mathcal{O}_F/\mathfrak{p}$.
2. The restriction of $\mathcal{C}_{\mathfrak{p}}$ to the ordinary locus $]X_K^{\mathrm{tor,ord}}[$ is the multiplicative part of $\mathcal{A}^{\mathrm{sa}}[\mathfrak{p}]$.
3. We equip $\mathcal{A}^{\mathrm{sa}}/\mathcal{C}_{\mathfrak{p}}$ with the induced action of $\mathcal{O}_F$, polarization and K^p-level structure. The quotient isogeny $\pi_{\mathfrak{p}}\colon \mathcal{A}^{\mathrm{sa}} \to \mathcal{A}^{\mathrm{sa}}/\mathcal{C}_{\mathfrak{p}}$ over $]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$ induces a finite flat map

$$\varphi_{\mathfrak{p}}\colon]X_K^{\mathrm{tor,ord}}[_{\underline{r}} \longrightarrow]X_K^{\mathrm{tor,ord}}[_{\underline{r}'} \tag{3.11.1}$$

such that $\varphi^*\mathcal{A}^{\mathrm{sa}} \simeq \mathcal{A}^{\mathrm{sa}}/\mathcal{C}_{\mathfrak{p}}$ together with all induced structures, where $\underline{r}' \in [0,p)^{\Sigma_p}$ is given by $r'_{\mathfrak{p}} = pr_{\mathfrak{p}}$ and $r'_{\mathfrak{q}} = r_{\mathfrak{q}}$ for $\mathfrak{q} \ne \mathfrak{p}$. The restriction of $\varphi_{\mathfrak{p}}$ to the non-cuspidal part $]X_K^{\mathrm{ord}}[_{\underline{r}}$ is finite étale of degree $N_{F/\mathbb{Q}}(\mathfrak{p})$.

In the sequel, for any point $(A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$ lying in the locus $]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$, we denote by $C_{\mathfrak{p}} \subset A[\mathfrak{p}]$ the $\mathfrak{p}$-canonical subgroup of A.

The isogeny $\pi_{\mathfrak{p}}$ induces a map on the relative de Rham cohomology

$$\pi_{\mathfrak{p}}^*\colon \varphi_{\mathfrak{p}}^*\mathcal{H}^1 = \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}^{\mathrm{sa},(\varphi_{\mathfrak{p}})}/]X_K^{\mathrm{tor,ord}}[_{\underline{r}}) \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}^{\mathrm{sa}}/]X_K^{\mathrm{tor,ord}}[_{\underline{r}}) = \mathcal{H}^1. \tag{3.11.2}$$

compatible with the Hodge filtration, the action of $\mathcal{O}_F$, and the connections ∇ on both sides.

3.12. Partial Frobenius $\mathrm{Fr}_{\mathfrak{p}}$. — Let $(\underline{k}, w)$ be a multiweight. The morphisms $\varphi_{\mathfrak{p}}$ and $\pi_{\mathfrak{p}}^* : \varphi_{\mathfrak{p}}^*(\mathcal{H}^1) \to \mathcal{H}^1$ induce a map of vector bundles on $]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$:

$$\pi_{\mathfrak{p}}^*\colon \varphi_{\mathfrak{p}}^*\mathscr{F}^{(\underline{k},w)} \to \mathscr{F}^{(\underline{k},w)}$$

compatible with all structures on both sides, and hence a morphism of de Rham complexes:

$$\pi_{\mathfrak{p}}^*\colon \mathrm{DR}_c^\bullet(\varphi_{\mathfrak{p}}^*\mathscr{F}^{(\underline{k},w)}) \to \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)}).$$

We define $\mathrm{Fr}_{\mathfrak{p}}$ to be the composite map on the cohomology group

$$\begin{array}{ccc} H^\star(]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}, \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) & \xrightarrow{\varphi_{\mathfrak{p}}^*} & H^\star(]X_K^{\mathrm{tor,ord}}[_{\underline{r}}, \mathrm{DR}_c^\bullet(\varphi_{\mathfrak{p}}^*\mathscr{F}^{(\underline{k},w)})) \\ & \searrow^{\mathrm{Fr}_{\mathfrak{p}}} & \downarrow{\pi_{\mathfrak{p}}^*} \\ & & H^\star(]X_K^{\mathrm{tor,ord}}[_{\underline{r}}, \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})). \end{array}$$

Taking direct limit as $\underline{r} \to 0^+$, one gets

$$\mathrm{Fr}_{\mathfrak{p}} \colon H^\star(\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})) \to H^\star(\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})). \tag{3.12.1}$$

We call $\mathrm{Fr}_{\mathfrak{p}}$ the *partial Frobenius* at $\mathfrak{p}$. To define the action of $\mathrm{Fr}_{\mathfrak{p}}$ on overconvergent cusp forms, we note that each $\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})e_J$ for $J \subseteq \Sigma_\infty$ appears as a direct summand of $\mathrm{Gr}_{\mathsf{F}}^\bullet \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$. Since the $\pi_{\mathfrak{p}}^*$-action on de Rham complexes is compatible with the F-filtrations, it induces a map

$$\pi_{\mathfrak{p}}^* \colon \pi_{\mathfrak{p}}^*\underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}) \to \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D}),$$

such that the resulting $\pi_{\mathfrak{p}}^* : \pi_{\mathfrak{p}}^*\mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)}) \to \mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$ is compatible with the quasi-isomorphism $\mathrm{BGG}_c^\bullet(\mathscr{F}^{(\underline{k},w)}) \hookrightarrow \mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)})$. Taking overconvergent sections, one gets a composite map

$$\begin{array}{ccc} H^0(]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}, \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})) & \xrightarrow{\varphi_{\mathfrak{p}}^*} & H^0(]X_K^{\mathrm{tor,ord}}[_{\underline{r}}, \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})) \\ & \searrow^{\mathrm{Fr}_{\mathfrak{p}}} & \downarrow{\pi_{\mathfrak{p}}^*} \\ & & H^0(]X_K^{\mathrm{tor,ord}}[_{\underline{r}}, \underline{\omega}^{\epsilon_J(\underline{k},w)} \otimes \underline{\Omega}^J(-\mathsf{D})) \end{array}$$

Letting $\underline{r} \to 0^+$, one gets the action of $\mathrm{Fr}_{\mathfrak{p}}$ on overconvergent cusp forms:

$$\mathrm{Fr}_{\mathfrak{p}} \colon S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp) \xrightarrow{\varphi_{\mathfrak{p}}^*} S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp) \xrightarrow{\pi_{\mathfrak{p}}^*} S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp).$$

By construction, these morphisms commute with the differential d^j in the complex $\mathscr{C}_K^\bullet$, and define an endomorphism of complexes $\mathrm{Fr}_{\mathfrak{p}} : \mathscr{C}_K^\bullet \to \mathscr{C}_K^\bullet$, which is compatible with the $\mathrm{Fr}_{\mathfrak{p}}$-action on $H^\star(\mathfrak{X}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger\mathrm{DR}_c^\bullet(\mathscr{F}^{(\underline{k},w)}))$ via Theorem 3.5.

3.13. Study of $\varphi_{\mathfrak{p}}$ over the ordinary locus. — The ordinary locus $]X_K^{\mathrm{tor,ord}}[$ is stable under $\varphi_{\mathfrak{p}}$. The restriction of $\varphi_{\mathfrak{p}}$ to $]X_K^{\mathrm{tor,ord}}[$ can be defined over the formal model $\mathfrak{X}_K^{\mathrm{tor,ord}}$. The morphism $\varphi_{\mathfrak{p}}$ induces a map on the differentials

$$\varphi_{\mathfrak{p}}^* : \varphi_{\mathfrak{p}}^*(\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})) \to \Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D}). \tag{3.13.1}$$

By (2.11.2), we have $\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D}) \simeq \bigoplus_{\tau\in\Sigma_\infty} \underline{\Omega}_\tau e_\tau$. For any $\mathfrak{q} \in \Sigma_p$, we put

$$\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})[\mathfrak{q}] = \bigoplus_{\tau\in\Sigma_{\infty/\mathfrak{q}}} \underline{\Omega}_\tau e_\tau.$$

This is the direct summand of $\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})$, where the action of $\mathcal{O}_F$ factors through $\mathcal{O}_{F_{\mathfrak{q}}}$. The action of $\varphi_{\mathfrak{p}}^*$ preserves $\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})[\mathfrak{q}]$ for all $\mathfrak{q} \in \Sigma_p$.

Lemma 3.14. — 1. *The action of $\varphi_{\mathfrak{p}}^*$ on $\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})[\mathfrak{q}]$ is an isomorphism for $\mathfrak{q} \neq \mathfrak{p}$. On $\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})[\mathfrak{p}]$, in a suitable local basis, the action of $\varphi_{\mathfrak{p}}^*$ is given by the multiplication by p.*

2. *If we regard* $\mathcal{O}_{\mathfrak{X}_K^{\mathrm{tor,ord}}}$ *as a finite flat algebra over* $\varphi_{\mathfrak{p}}^*(\mathcal{O}_{\mathfrak{X}_K^{\mathrm{tor,ord}}})$*, then we have*

$$\mathrm{Tr}_{\varphi_{\mathfrak{p}}}(\mathcal{O}_{\mathfrak{X}_K^{\mathrm{tor,ord}}}) \subseteq p^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}\varphi_{\mathfrak{p}}^*(\mathcal{O}_{\mathfrak{X}_K^{\mathrm{tor,ord}}}).$$

To prove this Lemma, we need some preliminary on the Serre-Tate local moduli.

Let $\overline{x} : \mathrm{Spec}(\overline{\mathbb{F}}_p) \to X_K^{\mathrm{ord}}$ be a geometric point in the ordinary locus, and $A_{\overline{x}}$ be the HBAV at $\overline{x}$. We denote by $\widehat{\mathcal{O}}_{\overline{x}}$ the completion of the local ring $\mathcal{O}_{\mathbf{X}^{\mathrm{tor,ord}}_{K,W(\overline{\mathbb{F}}_p)},\overline{x}}$ at $\overline{x}$. Let $\mathcal{D}ef_{\mathcal{O}_F}(A_{\overline{x}}[p^\infty])$ be the deformation space of $A_{\overline{x}}[p^\infty]$, i.e. the formal scheme over $W(\overline{\mathbb{F}}_p)$ that classifies the $\mathcal{O}_F$-deformations of $A_{\overline{x}}[p^\infty]$ to noetherian complete local $W(\overline{\mathbb{F}}_p)$-algebras with residue field $\overline{\mathbb{F}}_p$. By the Serre-Tate's theory, we have a canonical isomorphism of formal schemes

$$\mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{x}}) \cong \mathcal{D}ef_{\mathcal{O}_F}(A[p^\infty]). \tag{3.14.1}$$

The p-divisible group $A_{\overline{x}}[p^\infty]$ has a canonical decomposition

$$A_{\overline{x}}[p^\infty] = \prod_{\mathfrak{q}\in\Sigma_p} A_{\overline{x}}[\mathfrak{q}^\infty],$$

where each $A_{\overline{x}}[\mathfrak{q}^\infty]$ is an ordinary Barsotti-Tate $\mathcal{O}_{F_\mathfrak{q}}$-group of height 2 and dimension $f_\mathfrak{q} = [F_\mathfrak{q} : \mathbb{Q}_p]$. This induces a canonical decomposition of deformation spaces

$$\mathcal{D}ef_{\mathcal{O}_F}(A_{\overline{x}}[p^\infty]) \cong \prod_{\mathfrak{q}\in\Sigma_p} \mathcal{D}ef_{\mathcal{O}_{F_\mathfrak{q}}}(A_{\overline{x}}[\mathfrak{q}^\infty]), \tag{3.14.2}$$

where $\mathcal{D}ef_{\mathcal{O}_{F_\mathfrak{q}}}(A_{\overline{x}}[\mathfrak{q}^\infty])$ denotes the deformation space of $A_{\overline{x}}[\mathfrak{q}^\infty]$ as a Barsotti-Tate $\mathcal{O}_{F_\mathfrak{p}}$-modules, and the product is in the category of formal $W(\overline{\mathbb{F}}_p)$-schemes. Since $A_{\overline{x}}$ is ordinary, for each $\mathfrak{q} \in \Sigma_p$, we have a canonical exact sequence

$$0 \to A_{\overline{x}}[\mathfrak{q}^\infty]^\mu \to A_{\overline{x}}[\mathfrak{q}^\infty] \to A_{\overline{x}}[\mathfrak{q}]^{\mathrm{et}} \to 0,$$

where $A_{\overline{x}}[\mathfrak{q}^\infty]^\mu$ and $A_{\overline{x}}[\mathfrak{q}^\infty]^{\mathrm{et}}$ denote respectively the multiplicative part and the étale part of $A_{\overline{x}}[\mathfrak{q}^\infty]$. By Serre-Tate's theory, the deformation space $\mathcal{D}ef_{\mathcal{O}_{F_\mathfrak{q}}}(A_{\overline{x}}[\mathfrak{q}^\infty])$ has a natural formal group structure, and is canonically isomorphic to the formal group associated to the p-divisible

$$\mathrm{Hom}_{\mathcal{O}_{F_\mathfrak{q}}}(T_p(A_{\overline{x}}[\mathfrak{q}^\infty]^{\mathrm{et}}), A_{\overline{x}}[\mathfrak{q}^\infty]^\mu) \cong \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}_{F_\mathfrak{q}}.$$

Here, the last step used the fact that both $A_{\overline{x}}[\mathfrak{q}^\infty]^\mu$ and $A_{\overline{x}}[\mathfrak{q}^\infty]^{\mathrm{et}}$ has both height 1 as Barsotti-Tate $\mathcal{O}_{F_\mathfrak{q}}$-modules. Therefore, we have

$$\mathcal{D}ef_{\mathcal{O}_{F_\mathfrak{q}}}(A_{\overline{x}}[\mathfrak{q}^\infty]) \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{F_\mathfrak{q}} \simeq \widehat{\mathbb{G}}_m^{f_\mathfrak{q}}.$$

We choose an isomorphism

$$\mathcal{D}ef_{\mathcal{O}_\mathfrak{q}}(A_{\overline{x}}[\mathfrak{q}^\infty]) \simeq \mathrm{Spf}(W(\overline{\mathbb{F}}_p))[[t_{\mathfrak{q},1}, \dots, t_{\mathfrak{q},f_\mathfrak{p}}]], \tag{3.14.3}$$

so that the multiplication by p on $\mathcal{D}ef_{\mathcal{O}_{F_\mathfrak{q}}}(A_{\overline{x}}[\mathfrak{q}^\infty])$ is given by

$$[p](t_{\mathfrak{q},i}) = (1 + t_{\mathfrak{q},i})^p - 1.$$

Therefore, $\frac{dt_{\mathfrak{q},i}}{1+t_{\mathfrak{q},i}}(1 \leq i \leq f_\mathfrak{q})$ are invariant differentials, and they form a basis of $\widehat{\Omega}^1_{\mathcal{D}ef_{\mathcal{O}_{F_\mathfrak{q}}}(A_{\overline{x}}[\mathfrak{q}^\infty])/W(\overline{\mathbb{F}}_p)}$. By (3.14.1), we have

$$\widehat{\mathcal{O}}_{\overline{x}} \simeq W(\overline{\mathbb{F}}_p)[[\{t_{\mathfrak{q},i} : \mathfrak{q} \in \Sigma_p, 1 \leq i \leq f_\mathfrak{q}\}]].$$

The direct summand $\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\overline{x}}/W(\overline{\mathbb{F}}_p)}[\mathfrak{q}]$ of the differential module $\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\overline{x}}}$ is generated over $\widehat{\mathcal{O}}_{\overline{x}}$ by $\{\frac{dt_{\mathfrak{q},i}}{1+t_{\mathfrak{q},i}} : 1 \leq i \leq f_\mathfrak{q}\}$.

Proof of Lemma 3.14. — (1) Since $\pi_{\mathfrak{p}}: \mathcal{A} \to \mathcal{A}/\mathcal{C}_{\mathfrak{p}}$ induces an isomorphism on the $\mathfrak{q}$-divisible groups for $\mathfrak{q} \neq \mathfrak{p}$, $\varphi_{\mathfrak{p}}$ induces a canonical isomorphism between the local moduli:

$$\mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\overline{x}}[\mathfrak{p}^\infty]) \xrightarrow{\sim} \mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\varphi_{\mathfrak{q}}(\overline{x})}[\mathfrak{p}^\infty]).$$

It follows that $\varphi_{\mathfrak{p}}^*$ acts isomorphically on the direct summand $\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})[\mathfrak{q}]$ for $\mathfrak{q} \neq \mathfrak{p}$. To prove the second part of the Lemma, we take a geometric point $\overline{x}$ as above, and let $\varphi_{\mathfrak{p}}(\overline{x})$ be its image under $\varphi_{\mathfrak{p}}$. Let $\mathcal{A}_{\overline{x}}$ the base change of the universal HBAV to $\mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{x}})$. Then $\mathcal{A}_{\overline{x}}[\mathfrak{p}^\infty]$ is the universal deformation of $A_{\overline{x}}$ over $\mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\overline{x}}[\mathfrak{p}^\infty])$. It is an ordinary Barsotti-Tate $\mathcal{O}_{F_{\mathfrak{p}}}$-modules, i.e. an extension of its étale part by its multiplicative part. The isogeny $\pi_{\mathfrak{p}}: \mathcal{A}_{\overline{x}} \to \mathcal{A}_{\varphi_{\mathfrak{p}}(\overline{x})} = \mathcal{A}_{\overline{x}}/\mathcal{C}_{\mathfrak{p},\overline{x}}$ induces an exact sequence of $\mathfrak{p}$-divisible groups

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_{\overline{x}}[\mathfrak{p}^\infty]^\mu & \to & \mathcal{A}_{\overline{x}}[\mathfrak{q}^\infty] & \to & \mathcal{A}_{\overline{x}}[\mathfrak{p}^\infty]^{\mathrm{et}} & \to & 0 \\ & & \downarrow{\scriptstyle \pi_{\mathfrak{p}}^\mu} & & \downarrow{\scriptstyle \pi_{\mathfrak{p}}} & & \downarrow{\scriptstyle \pi_{\mathfrak{p}}^{\mathrm{et}}} & & \\ 0 & \to & \mathcal{A}_{\varphi_{\mathfrak{p}}(\overline{x})}[\mathfrak{p}^\infty]^\mu & \to & \mathcal{A}_{\varphi_{\mathfrak{p}}(\overline{x})}[\mathfrak{p}^\infty] & \to & \mathcal{A}_{\varphi_{\mathfrak{p}}(\overline{x})}[\mathfrak{p}^\infty]^{\mathrm{et}} & \to & 0 \end{array}$$

Since the $\mathfrak{p}$-canonical subgroup $\mathcal{C}_{\mathfrak{p},\overline{x}}$ coincides with the p-torsion of $\mathcal{A}_{\overline{x}}[\mathfrak{p}^\infty]^\mu$, the isogeny $\pi_{\mathfrak{p}}^\mu$ is identified with the multiplication by p up to isomorphism, and $\pi_{\mathfrak{p}}^{\mathrm{et}}$ is an isomorphism. This implies that, there exists an isomorphism

$$\phi\colon \mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\overline{x}}[\mathfrak{p}^\infty]) \to \mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\varphi(\overline{x})}[\mathfrak{p}^\infty])$$

such that $\varphi_{\mathfrak{p}} = p \cdot \phi$. Hence, the map induced by $\varphi_{\mathfrak{p}}$ on the invariant differentials on $\mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\varphi(\overline{x})}[\mathfrak{p}^\infty])$ is given by the multiplication by p composed with the isomorphism induced by ϕ. The Lemma follows from the fact that $\Omega^1_{\mathfrak{X}_K^{\mathrm{tor,ord}}}(\log \mathsf{D})[\mathfrak{p}]$ around $\varphi_{\mathfrak{p}}(\overline{x})$ is generated by the invariant differentials of $\mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\varphi_{\mathfrak{p}}(\overline{x})}[\mathfrak{p}^\infty])$ via the isomorphism (3.14.1).

(2) The problem is local. Let $\overline{x}$ and $\varphi_{\mathfrak{p}}(\overline{x})$ as above. It suffices to show that

$$\mathrm{Tr}_{\varphi_{\mathfrak{p}}^*}(\mathcal{O}_{\overline{x}}) \subseteq p^{f_{\mathfrak{p}}}\varphi_{\mathfrak{p}}^*(\mathcal{O}_{\varphi_{\mathfrak{p}}(\overline{x})})$$

We always use (3.14.1) to identify $\mathrm{Spf}(\mathcal{O}_{\overline{x}})$ with $\mathcal{D}ef_{\mathcal{O}_F}(A_{\overline{x}}[p^\infty])$. Let $\varphi'_{\mathfrak{p}}$ denote the endomorphism on $\mathcal{D}ef_{\mathcal{O}_F}(A_{\overline{x}}[p^\infty])$ that gives the multiplication by p on $\mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\overline{x}}[\mathfrak{p}^\infty])$ and the identity on $\mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{q}}}}(A_{\overline{x}}[\mathfrak{q}^\infty])$ with $\mathfrak{q} \neq \mathfrak{p}$. By the discussion above, there exists an isomorphism $\phi: \mathcal{D}ef_{\mathcal{O}_F}(A_{\overline{x}}[p^\infty]) \to \mathcal{D}ef_{\mathcal{O}_F}(A_{\overline{x}}[p^\infty])$ such that $\varphi_{\mathfrak{p}} = \phi \circ \varphi'_{\mathfrak{p}}$. Thus it suffices to prove that $\mathrm{Tr}_{\varphi'^*_{\mathfrak{p}}}$ is divisible by $p^{f_{\mathfrak{p}}}$. Then we may further reduce the problem to showing that the trace map of the multiplication by p on $\mathcal{D}ef_{\mathcal{O}_{F_{\mathfrak{p}}}}(A_{\overline{x}}[\mathfrak{p}^\infty])$ is divisible by $p^{f_{\mathfrak{p}}}$. This follows from an easy computation using the canonical coordinates $\{t_{\mathfrak{p},i} : 1 \leq i \leq f_{\mathfrak{p}}\}$ in (3.14.3). □

3.15. $U_{\mathfrak{p}}$-correspondence. — Let $\underline{r} = (r_{\mathfrak{q}})_{\mathfrak{q}} \in ((0,p) \cap \mathbb{Q})^{\Sigma_p}$ be a tuple with $r_{\mathfrak{p}} < 1$ as in Subsection 3.11, and $\underline{r}' = (r'_{\mathfrak{q}})_{\mathfrak{q}} \in [0,p)^{\Sigma_p}$ be such that $r'_{\mathfrak{p}} = p\, r_{\mathfrak{p}}$ and $r'_{\mathfrak{q}} = r_{\mathfrak{q}}$ with $\mathfrak{q} \neq \mathfrak{p}$. Let $\mathcal{A}^{\mathrm{sa}}$ be the family of abelian schemes over $]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$, and $]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}$ be the rigid analytic space that classifies the $\mathcal{O}_F$-stable finite flat group schemes $\mathcal{H} \subseteq \mathcal{A}^{\mathrm{sa}}[\mathfrak{p}]$ which is disjoint with the $\mathfrak{p}$-canonical subgroup $\mathcal{C}_{\mathfrak{p}}$, i.e. outside the toroidal boundary, $]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}$ parametrizes the tuples $(A, \iota, \lambda, \alpha_{K^p}, H)$

- $(A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$ is a point of $]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$,
- $H \subset A[\mathfrak{p}]$ is a subgroup stable under $\mathcal{O}_F$, étale locally isomorphic to $\mathcal{O}_F/\mathfrak{p}$ and disjoint with the $\mathfrak{p}$-canonical subgroup $C_{\mathfrak{p}} \subset A[\mathfrak{p}]$.

We have two projections

$$\begin{array}{ccc} &]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'} & \\ {\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} \\ ]X_K^{\mathrm{tor,ord}}[_{\underline{r}'} & &]X_K^{\mathrm{tor,ord}}[_{\underline{r}}, \end{array} \tag{3.15.1}$$

whose effect on non-cuspidal points are given by

$$\mathrm{pr}_1(A,\iota,\bar{\lambda},\bar{\alpha}_{K^p},H)\mapsto(A,\iota,\bar{\lambda},\bar{\alpha}_{K^p})$$
$$\mathrm{pr}_2(A,\iota,\bar{\lambda},\bar{\alpha}_{K^p},H)\mapsto(A/H,\iota',\bar{\lambda}',\bar{\alpha}'_{K^p}).$$

Here, $(A/H,\iota',\bar{\lambda}',\bar{\alpha}'_{K^p})$ denotes the quotient rigid analytic HBAV A/H with the induced polarization and K^p-level structure. In the terminology of [**GK09**], the subgroups H are *anti-canonical* at $\mathfrak{p}$, and [**GK09**, 5.4.3] implies that image of pr_2 lies in $]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$.

Lemma 3.16. — *The morphism* pr_1 *is finite étale of degree* $N_{F/\mathbb{Q}}(\mathfrak{p})$. *The map* pr_2 *is an isomorphism of rigid analytic spaces, with the inverse map* $\tilde{S}_{\mathfrak{p}^{-1}}\circ\tilde{\varphi}_{\mathfrak{p}}$, *where*

$$\tilde{\varphi}_{\mathfrak{p}}\colon(A,\iota',\bar{\lambda},\bar{\alpha}_{K^p})\mapsto(A/C_{\mathfrak{p}},\bar{\iota},\bar{\lambda}',\bar{\alpha}'_{K^p},A[\mathfrak{p}]/C_{\mathfrak{p}})$$
$$\tilde{S}_{\mathfrak{p}^{-1}}=\tilde{S}_{\mathfrak{p}}^{-1}\colon(A,\iota,\bar{\lambda},\bar{\alpha}_{K^p},H)\mapsto(A\otimes_{\mathcal{O}_F}\mathfrak{p},\iota'',\bar{\lambda}'',\bar{\alpha}''_{K^p},H\otimes_{\mathcal{O}_F}\mathfrak{p}).$$

Here, $(\iota',\bar{\lambda}',\bar{\alpha}'_{K^p})$ *and* $(\iota'',\bar{\lambda}'',\bar{\alpha}''_{K^p})$ *denote the natural induced structures on the corresponding objects. In particular, we have*

$$\mathrm{pr}_1=S_{\mathfrak{p}^{-1}}\circ\varphi_{\mathfrak{p}}\circ\mathrm{pr}_2, \tag{3.16.1}$$

where $\varphi_{\mathfrak{p}}$ *is defined in* (3.11.1), *and by abuse of notation,* $S_{\mathfrak{p}^{-1}}=S_{\mathfrak{p}}^{-1}$ *denotes the automorphism on* $]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$ *given by* $(A,\iota,\lambda,\alpha_{K^p})\mapsto(A\otimes_{\mathcal{O}_F}\mathfrak{p},\iota'',\bar{\lambda}'',\bar{\alpha}''_{K^p})$.

Proof. — The statement for pr_1 is clearly. To see pr_2 is an isomorphism, we take a point $(A,H):=(A,\iota,\bar{\lambda},\bar{\alpha}_{K^p},H)$ in $]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}$. We have $\mathrm{pr}_2(A,H)=A/H$, and $A[\mathfrak{p}]/H$ is the $\mathfrak{p}$-canonical subgroup of $A'=A/H$. So we have

$$\tilde{\varphi}_{\mathfrak{p}}(A')=(A'/(A[\mathfrak{p}]/H),A'[\mathfrak{p}]/(A[\mathfrak{p}]/H))=(A/A[\mathfrak{p}],\bar{H})=(A\otimes_{\mathcal{O}_F}\mathfrak{p}^{-1},H\otimes_{\mathcal{O}_F}\mathfrak{p}^{-1}).$$

with all the induced structures. The Lemma now follows immediately. ☐

We put $\mathsf{D}^{\mathfrak{p}}=\mathrm{pr}_2^{-1}\mathsf{D}$. Then the correspondence (3.15.1) induces isomorphisms of differentials

$$\mathrm{pr}_1^*\colon\mathrm{pr}_1^*\Omega^1_{]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}}(\log\mathsf{D})\xrightarrow{\simeq}\Omega^1_{]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}}(\log\mathsf{D}^{\mathfrak{p}})$$
$$\mathrm{pr}_2^*\colon\mathrm{pr}_2^*\Omega^1_{]X_K^{\mathrm{tor,ord}}[_{\underline{r}}}(\log\mathsf{D})\xrightarrow{\simeq}\Omega^1_{]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}}(\log\mathsf{D}^{\mathfrak{p}}),$$

which preserves the natural action of $\mathcal{O}_F$ on both sides induced from the extended Kodaira-Spencer isomorphism (2.11.2). In particular, for each subset $J\subseteq\Sigma_\infty$, these isomorphisms induce an isomorphism

$$\phi_{12}\colon\mathrm{pr}_2^*\underline{\Omega}^Je_J\xrightarrow{\simeq}\mathrm{pr}_1^*\underline{\Omega}^Je_J. \tag{3.16.2}$$

3.17. Norms. — We recall the construction of p-adic norms on rigid analytic varieties. Suppose we are given an admissible formal scheme $\mathfrak{Z}$ over $\mathcal{O}_{L_\wp}$, and a vector bundle $\mathscr{E}$ on $\mathfrak{Z}$. Let $\mathfrak{Z}_{\mathrm{rig}}$ denote the rigid analytic space over $L_\wp$ associated to $\mathfrak{Z}$, and $\mathscr{E}_{\mathrm{rig}}$ be the associated vector bundle on $\mathfrak{Z}_{\mathrm{rig}}$. We denote by $|\cdot|$ the non-archimedean norm on $\mathbb{C}_p$ normalized by $|p|=p^{-1}$. For a quasi-compact open subset $U\subseteq\mathfrak{Z}_{\mathrm{rig}}$, one can define a norm $\|\cdot\|_U$ on $\Gamma(U,\mathscr{E}_{\mathrm{rig}})$ such that $\|\lambda\cdot s\|_U=|\lambda|\cdot\|s\|_U$ for $\lambda\in\mathbb{C}_p$ and $s\in\Gamma(U,\mathscr{E}_{\mathrm{rig}})$ as follows. Recall that a point $x\in\mathfrak{Z}_{\mathrm{rig}}$ defined over an extension $L'_{\wp'}/L_\wp$ is equivalent to a morphism of $\mathcal{O}_{L_\wp}$-formal schemes $x:\mathrm{Spf}(\mathcal{O}_{L'_\wp})\to\mathfrak{Z}$. Given a section $s\in\Gamma(U,\mathscr{E}_{\mathrm{rig}})$ and a point $x\in U$ defined over

$L'_{\wp'}/L_\wp$, we denote by $x^*(s) \in \mathscr{E}_x \otimes_{\mathcal{O}_{\wp'}} L'_{\wp'}$ the inverse image of s by x. We define $|s(x)|$ to be the maximum of $|\lambda|$ where $\lambda \in L'_{\wp'}$ such that $\lambda \cdot s \in \mathscr{E}_x$, and put

$$\|s\|_U = \max_{x\in U} |s(x)|.$$

We apply the construction above to the integral model $\mathfrak{X}_K^{\mathrm{tor}}$ and $\underline{\Omega}^J$ over it. For a quasi-compact admissible open subset $U \subset]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$, we have a well defined norm $\|\cdot\|_U$ on the space of sections $\Gamma(U, \underline{\Omega}^J)$. For a section s of $\Omega^1_{]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}}(\log \mathsf{D}^{\mathfrak{p}})$ over a quasi-compact subset $V \subseteq]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}$, we put $\|s\|_V = \|(\mathrm{pr}_2^*)^{-1}(s)\|_{\mathrm{pr}_2(V)}$.

Lemma 3.18. — *Let s be a local section of $\underline{\Omega}^J e_J$ defined over a quasi-compact admissible open subset U contained in the ordinary locus $]\mathfrak{X}_K^{\mathrm{tor,ord}}[$. We have*

$$\|\mathrm{pr}_1^*(s)\|_{\mathrm{pr}_1^{-1}(U)} = p^{-\#(J\cap\Sigma_{\infty/\mathfrak{p}})}\|s\|_U.$$

Proof. — We can easily reduce to the case where $J = \{\tau\}$ contains only one element. We have to show that

$$\|\mathrm{pr}_1^*(s)\|_{\mathrm{pr}_1^{-1}(U)} = \begin{cases} \|s\|_U & \text{if } \tau \notin \Sigma_{\infty/\mathfrak{p}};\\ p^{-1}\|s\|_U & \text{if } \tau \in \Sigma_{\infty/\mathfrak{p}}.\end{cases}$$

By definition, we have $\|\mathrm{pr}_1^*(s)\|_{\mathrm{pr}_1^{-1}(U)} = \|(\mathrm{pr}_2^*)^{-1}\mathrm{pr}_1^*(s)\|_{\mathrm{pr}_2(\mathrm{pr}_1^{-1}(U))}$. By (3.16.1), we have

$$(\mathrm{pr}_2^*)^{-1}\mathrm{pr}_1^* = (\mathrm{pr}_2^*)^{-1} \circ (\mathrm{pr}_2^* \circ \varphi_{\mathfrak{p}}^* \circ S^*_{\mathfrak{p}^{-1}}) = \varphi_{\mathfrak{p}}^* \circ S^*_{\mathfrak{p}^{-1}}.$$

It follows from Lemma 3.14 that

$$\|\varphi_{\mathfrak{p}}^*(s)\|_{\mathrm{pr}_2(\mathrm{pr}_1^{-1}(U))} = \begin{cases} \|s\|_U & \text{if } \tau \notin \Sigma_{\infty/\mathfrak{p}}\\ p^{-1}\|s\|_U & \text{if } \tau \in \Sigma_{\infty/\mathfrak{p}}.\end{cases}$$

Since $S_{\mathfrak{p}^{-1}}$ is an automorphism defined over the integral model $\mathfrak{X}_K^{\mathrm{tor}}$, $S^*_{\mathfrak{p}^{-1}}$ has norm 1. This concludes the proof. □

We have an isogeny of semi-abelian schemes over $]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}$:

$$\breve{\pi}_{\mathfrak{p}} \colon \mathrm{pr}_1^*\mathcal{A}^{\mathrm{sa}} \to \mathrm{pr}_2^*\mathcal{A}^{\mathrm{sa}} = \mathcal{A}^{\mathrm{sa}}/\mathcal{H},$$

whose $\mathcal{H} \subset \mathcal{A}^{\mathrm{sa}}[\mathfrak{p}]$ is the tautological subgroup scheme disjoint from $\mathcal{C}_{\mathfrak{p}}$. It induces a morphism on the relative de Rham cohomology

$$\breve{\pi}_{\mathfrak{p}}^* \colon \mathrm{pr}_2^*\mathcal{H}^1 \to \mathrm{pr}_1^*\mathcal{H}^1$$

compatible with all the structures on both sides. In particular, for each $\tau \in \Sigma_\infty$, it induces a morphism $\breve{\pi}_{\mathfrak{p}} \colon \mathcal{H}^1_\tau \to \mathcal{H}^1_\tau$ compatible with the Hodge filtration $0 \to \underline{\omega}_\tau \to \mathcal{H}^1_\tau \to \mathrm{Lie}((\mathcal{A}^{\mathrm{sa}})^\vee)_\tau \to 0$.

Lemma 3.19. — *Let $x = (A, \iota, \lambda, \alpha_{K^p}, H)$ be a rigid point in $]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}$ defined over the ring of integers $\mathcal{O}_{\wp'}$ of a finite extension $L'_{\wp'}/L_\wp$, and $\breve{\pi}_{\mathfrak{p},x} : A \to A' = A/H$ be the canonical isogeny. Assume that A has ordinary good reduction. Let ω_τ and η_τ (resp. ω'_τ and η'_τ) be a basis of $\mathcal{H}^1_\tau(A/\mathcal{O}_{\wp'})$ (resp. $\mathcal{H}^1_\tau(A'/\mathcal{O}_{\wp'})$) over $\mathcal{O}_{wp'}$ adapted to the Hodge filtration, and write*

$$\breve{\pi}^*_{\mathfrak{p},x}(\omega'_\tau) = a_\tau\omega_\tau, \quad \breve{\pi}^*_{\mathfrak{p},x}(\eta'_\tau) \equiv b_\tau\eta_\tau \mod \underline{\omega}_\tau.$$

Then we have $\mathrm{val}_p(a_\tau) = 0$ for all $\tau \in \Sigma_\infty$, $\mathrm{val}_p(b_\tau) = 0$ if $\tau \notin \Sigma_{\infty/\mathfrak{p}}$ and $\mathrm{val}_p(b_\tau) = 1$ if $\tau \in \Sigma_{\infty/\mathfrak{p}}$. In particular, we have

$$\breve{\pi}^*_{\mathfrak{p},x}(\omega'_\tau \wedge \eta'_\tau) = a_\tau b_\tau \omega_\tau \wedge \eta_\tau,$$

with $\mathrm{val}_p(a_\tau b_\tau) = 0$ if $\tau \notin \Sigma_{\infty/\mathfrak{p}}$ and $\mathrm{val}_p(a_\tau b_\tau) = 1$ if $\tau \in \Sigma_{\infty/\mathfrak{p}}$.

Proof. — The problem depends only on the p-divisible group $A[\mathfrak{p}^\infty]$. The isogeny $\breve{\pi}_{\mathfrak{p}}$ induces an isomorphism of the p-divisible groups $A[\mathfrak{q}^\infty] \xrightarrow{\sim} A'[\mathfrak{q}^\infty]$ over $\mathcal{O}_{\wp'}$ for $\mathfrak{q} \neq \mathfrak{p}$. Thus, the statements for $\tau \notin \Sigma_{\infty/\mathfrak{p}}$ are evident. The subgroup $H \subset A[\mathfrak{p}]$ with $H \neq C_{\mathfrak{p}}$ is necessarily étale, since A has good ordinary reduction. Therefore, $\breve{\pi}_{\mathfrak{p},x}$ is étale and induces an isomorphism

$$\underline{\omega}_{A'[\mathfrak{p}^\infty]} = \bigoplus_{\tau \in \Sigma_{\infty/\mathfrak{p}}} \omega_{A',\tau} \xrightarrow{\sim} \bigoplus_{\tau \in \Sigma_{\infty/\mathfrak{p}}} \underline{\omega}_{A,\tau} = \underline{\omega}_{A[\mathfrak{p}^\infty]}.$$

It follows immediately that a_τ are units in $\mathcal{O}_{\wp'}$ for $\tau \in \Sigma_{\infty/\mathfrak{p}}$. To show that $\mathrm{val}_p(b_\tau) = 1$, we consider the dual isogeny $\breve{\pi}^\vee_{\mathfrak{p},x} \colon A'^\vee \to A^\vee$. Let $A'^\vee[\mathfrak{p}^\infty]^\mu$ and $A^\vee[\mathfrak{p}^\infty]^\mu$ be respectively the multiplicative part of $A'^\vee[\mathfrak{p}^\infty]$ and $A^\vee[\mathfrak{p}^\infty]$. We have an induced isogeny

$$(\breve{\pi}^\vee_{\mathfrak{q},x})^\mu : A'^\vee[\mathfrak{p}^\infty]^\mu \to A^\vee[\mathfrak{p}^\infty]^\mu.$$

The kernels of $\breve{\pi}^\vee_{\mathfrak{p},x}$ and $(\breve{\pi}^\vee_{\mathfrak{p},x})^\mu$ are both $H^\vee$, which is identified with the p-torsion of $A'^\vee[\mathfrak{p}^\infty]^\mu$ (since $\mathfrak{p}$ is unramified). Hence, the induced map on $\mathrm{Lie}(A'^\vee[\mathfrak{p}^\infty]^\mu) \to \mathrm{Lie}(A^\vee[\mathfrak{p}^\infty]^\mu)$ is given by the multiplication by p up to units, whence $\mathrm{val}_p(b_\tau) = 1$ for all $\tau \in \Sigma_{\infty/\mathfrak{p}}$. Now the Lemma follows from the fact that $\mathrm{Lie}(A'^\vee)_\tau = \mathrm{Lie}(A'^\vee[\mathfrak{p}^\infty]^\mu)_\tau$ for $\tau \in \Sigma_{\infty/\mathfrak{q}}$, since A' is ordinary. □

3.20. $U_{\mathfrak{p}}$-operator. — We now define the $U_{\mathfrak{p}}$-operator on $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ and on the complex $\mathscr{C}^\bullet_K$. The map $\breve{\pi}^*_{\mathfrak{p}} : \mathrm{pr}_2^*\mathcal{H}^1 \to \mathrm{pr}_1^*\mathcal{H}^1$ induces a map $\breve{\pi}^*_{\mathfrak{p}} : \mathrm{pr}_2^*\mathscr{F}^{(\underline{k},w)} \to \mathrm{pr}_1^*\mathscr{F}^{(\underline{k},w)}$ and hence a map of de Rham complexes

$$\breve{\pi}^*_{\mathfrak{p}} \colon \mathrm{DR}^\bullet_c(\mathrm{pr}_2^*\mathscr{F}^{(\underline{k},w)}) \to \mathrm{DR}^\bullet_c(\mathrm{pr}_1^*\mathscr{F}^{(\underline{k},w)}) \tag{3.20.1}$$

compatible with the $\mathtt{F}$-filtrations on both sides defined in Subsection 2.14.

We define $U_{\mathfrak{p}}$-operator to be the composite map on the cohomology groups

$$\begin{array}{ccc} H^\star(]X_K^{\mathrm{tor,ord}}[_{\underline{r}}, \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})) & \overset{U_{\mathfrak{p}}}{\dashrightarrow} & H^\star(]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}, \mathrm{DR}^\bullet_c(\mathscr{F}^{\underline{k},w})) \\ \downarrow{\scriptstyle \mathrm{pr}_2^*} & & \uparrow{\scriptstyle \mathrm{Tr}_{\mathrm{pr}_1}} \\ H^\star(]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}, \mathrm{DR}^\bullet_c(\mathrm{pr}_2^*\mathscr{F}^{(\underline{k},w)})) & \xrightarrow{\breve{\pi}^*_{\mathfrak{p}}} & H^\star(]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}'}, \mathrm{DR}^\bullet_c(\mathrm{pr}_1^*\mathscr{F}^{(\underline{k},w)})) \end{array},$$

where the existence of the trace map $\mathrm{Tr}_{\mathrm{pr}_1}$ follows from similar arguments as in Corollary 3.9.

By letting $\underline{r} \to 0^+$ (so $\underline{r}' \to 0^+$ as well), we get a map

$$U_{\mathfrak{p}} \colon H^\star(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})) \to H^\star(\mathfrak{X}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})). \tag{3.20.2}$$

We now define the $U_{\mathfrak{p}}$-action on the complex $\mathscr{C}^\bullet_K$ which is compatible with that on $H^\star_{\mathrm{rig}}(\mathrm{Sh}^{\mathrm{tor,ord}}_{k_0}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ by Theorem 3.5.

We come back to the map (3.20.1). By the canonical isomorphic embedding $\mathrm{BGG}^\bullet_c(\mathscr{F}^{(\underline{k},w)}) \hookrightarrow \mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$, $\breve{\pi}^*_{\mathfrak{p}}$ on the de Rham complex induces a commutative diagram

$$\begin{array}{ccc} \mathrm{BGG}^\bullet_c(\mathrm{pr}_2^*\mathscr{F}^{(\underline{k},w)}) & \xrightarrow{\breve{\pi}^*_{\mathfrak{p}}} & \mathrm{BGG}^\bullet_c(\mathrm{pr}_1^*\mathscr{F}^{(\underline{k},w)}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{DR}^\bullet_c(\mathrm{pr}_2^*(\mathscr{F}^{(\underline{k},w)})) & \xrightarrow{\breve{\pi}^*_{\mathfrak{p}}} & \mathrm{DR}^\bullet_c(\mathrm{pr}_1^*\mathscr{F}^{(\underline{k},w)}). \end{array} \tag{3.20.3}$$

Note that $\mathrm{BGG}^j_c(\mathrm{pr}_i^*\mathscr{F}^{(\underline{k},w)}) = \mathrm{pr}_i^*\mathrm{BGG}^j_c(\mathscr{F}^{(\underline{k},w)})$ for $i = 1, 2$ and $0 \leq j \leq g$ by (2.12.1). Explicitly, the induced map by $\breve{\pi}^*_{\mathfrak{p}}$ on the each term of the complex $\mathrm{BGG}^\bullet_c(\mathrm{pr}_2^*\mathscr{F}^{(\underline{k},w)})$ has the following description. Since (3.20.1) is compatible with the $\mathtt{F}$-filtrations, it induces, for each $n \in \mathbb{Z}$, a map

$$\breve{\pi}^*_{\mathfrak{p}} \colon \mathrm{Gr}^n_{\mathtt{F}}(\mathrm{DR}^\bullet_c(\mathrm{pr}_2^*\mathscr{F}^{(\underline{k},w)})) \to \mathrm{Gr}^n_{\mathtt{F}}(\mathrm{DR}^\bullet_c(\mathrm{pr}_1^*\mathscr{F}^{(\underline{k},w)})).$$

By Theorem 2.16 and (2.15.4), we have

$$\mathrm{Gr}^n_{\mathsf{F}}(\mathrm{DR}^\bullet_c(\mathrm{pr}_i^*\mathscr{F}^{(\underline{k},w)}))\cong \mathrm{Gr}^n_{\mathsf{F}}\mathrm{BGG}^\bullet_c(\mathrm{pr}_i^*\mathscr{F}^{(\underline{k},w)})\cong \bigoplus_{\substack{J\subseteq\Sigma_\infty\\ n_J=n}}\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\underline{\Omega}^J e_J[-\#J],$$

where $n_J:=\sum_{\tau\in J}(k_\tau-1)+\sum_{\tau\in\Sigma_R}\frac{w-k_\tau}{2}$. This induces, for each subset $J\subseteq\Sigma_\infty$, an $\mathcal{O}_{]X_K^{\mathrm{tor,ord}}[^{\mathfrak{p}}_{\underline{r}}}$-linear map on the J-component:

$$\check{\pi}^*_{\mathfrak{p}}|_{\underline{\omega}^{\epsilon_J(\underline{k},w)}}\colon \mathrm{pr}_2^*\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\mathrm{pr}_2^*\underline{\Omega}^J(-\mathsf{D}^{\mathfrak{p}})\xrightarrow{\check{\pi}^*_{\mathfrak{p}}\otimes\phi_{12}}\mathrm{pr}_1^*\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\mathrm{pr}_1^*\underline{\Omega}^J(-\mathsf{D}^{\mathfrak{p}}),$$

where ϕ_{12} is the isomorphism defined in (3.16.2). By putting the J-summands with $|J|=j$ together, we get

$$\check{\pi}^*_{\mathfrak{p}}|_{\mathrm{BGG}^j_c}\colon \mathrm{pr}_2^*\mathrm{BGG}^j_c(\mathscr{F}^{(\underline{k},w)})\to\mathrm{pr}_1^*\mathrm{BGG}^j_c(\mathscr{F}^{(\underline{k},w)}).$$

induced from the diagram (3.20.3).

Let $U_1\subset]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$ and $U_2\subset]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$ be quasi-compact admissible open subsets such that $\mathrm{pr}_1^{-1}(U_1)\subset\mathrm{pr}_2^{-1}(U_2)$. We denote by $\mathrm{res}_{12}:\mathrm{pr}_2^{-1}(U_2)\to\mathrm{pr}_1^{-1}(U_1)$ the natural restriction map. For every $J\subseteq\Sigma_\infty$, we have a composite map $U_{\mathfrak{p}}$

$$\begin{array}{ccc}\Gamma(U_2,\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\underline{\Omega}^J(-\mathsf{D})) & \overset{U_{\mathfrak{p}}}{\dashrightarrow} & \Gamma(U_1,\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\underline{\Omega}^J(-\mathsf{D}))\\ \downarrow{\scriptstyle \mathrm{pr}_2^*} & & \uparrow{\scriptstyle \mathrm{Tr}_{\mathrm{pr}_1}}\\ \Gamma(\mathrm{pr}_2^{-1}(U_2),\mathrm{pr}_2^*\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\underline{\Omega}^J(-\mathsf{D}^{\mathfrak{p}})) & \xrightarrow{\check{\pi}^*_{\mathfrak{p}}\circ\mathrm{res}_{12}} & \Gamma(\mathrm{pr}_1^{-1}(U_1),\mathrm{pr}_1^*\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\underline{\Omega}^J(-\mathsf{D}^{\mathfrak{p}}))\end{array}$$

Taking $U_1=]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$ and $U_2=]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$ and making $\underline{r}\to0^+$, one gets an endomorphism

$$U_{\mathfrak{p}}\colon S^\dagger_{\epsilon_J(\underline{k},w)}(K,L_\wp)\to S^\dagger_{\epsilon_J(\underline{k},w)}(K,L_\wp).$$

Since $]X_K^{\mathrm{tor,ord}}[_{\underline{r}'}$ is a strict neighborhood of $]X_K^{\mathrm{tor,ord}}[_{\underline{r}}$, the endomorphism $U_{\mathfrak{p}}$ on $S^\dagger_{\epsilon_J(\underline{k},w)}(K,L_\wp)$ is completely continuous. As in the case of $\Phi_{\mathfrak{p}}$, $U_{\mathfrak{p}}$ commutes with the differential d^j of the complex $\mathscr{C}^\bullet_K$ by its very construction. By putting all $J\subseteq\Sigma_\infty$ together, one obtains actually an endomorphism of complexes $U_{\mathfrak{p}}\colon\mathscr{C}^\bullet_K\to\mathscr{C}^\bullet_K$. By our construction, it is clear that the $U_{\mathfrak{p}}$ on $H^\star(\mathscr{C}^\bullet_K)$ is canonically identified with the one defined in (3.20.2) via Theorem 3.5.

Remark 3.21. — On $S^\dagger_{(\underline{k},w)}(K,L_\wp)$, our definition $U_{\mathfrak{p}}$-operator coincides with the usual $U_{\mathfrak{p}}$-operator defined as in [**KL05**] i.e. it induces the classical normalized $U_{\mathfrak{p}}$-operator on classical forms $S_{(\underline{k},w)}(K^p\mathrm{Iw}_p,L_\wp)$. However, for $J\subset\Sigma_\infty$ with $\Sigma_{\infty/\mathfrak{p}}\cap J\neq\Sigma_{\infty/\mathfrak{p}}$, our $U_{\mathfrak{p}}$-operator on $S^\dagger_{\epsilon_J(\underline{k},w)}(K,L_\wp)$, which is induced from the action of $U_{\mathfrak{p}}$-correspondence on $\mathrm{DR}^\bullet_c(\mathscr{F}^{(\underline{k},w)})$, is not the usual $U_{\mathfrak{p}}$ studied by Kisin-Lai [**KL05**]. Actually, it is easy to check that Kisin-Lai's $U_{\mathfrak{p}}$-operators does not commute with $d^j:\mathscr{C}^j_K\to\mathscr{C}^{j+1}_K$. If p is inert in F, our definition of $U_{\mathfrak{p}}$ on $S^\dagger_{\epsilon_J(\underline{k},w)}(K,L_\wp)$ coincides with $p^{\sum_{\tau\notin J}(k_\tau-1)}$ times Kisin-Lai's $U_{\mathfrak{p}}$.

There exists a simple relationship between the partial Frobenius $\mathrm{Fr}_{\mathfrak{p}}$ and the operator $U_{\mathfrak{p}}$:

Lemma 3.22. — *As operators on the cohomology groups* $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})$ *or on* $\mathscr{C}^\bullet_K$, *we have*

$$U_{\mathfrak{p}}\mathrm{Fr}_{\mathfrak{p}}=N_{F/\mathbb{Q}}(\mathfrak{p})S_{\mathfrak{p}},$$

where the action of $S_{\mathfrak{p}}$ *is defined in* (3.10.1).

Proof. — By the definition, we have

$$U_{\mathfrak{p}}\mathrm{Fr}_{\mathfrak{p}}=\mathrm{Tr}_{\mathrm{pr}_1}\circ\check{\pi}^*_{\mathfrak{p}}\circ\mathrm{pr}_2^*\circ\pi^*_{\mathfrak{p}}\circ\varphi^*_{\mathfrak{p}}=\mathrm{Tr}_{\mathrm{pr}_1}\circ\check{\pi}^*_{\mathfrak{p}}\circ\pi^*_{\mathfrak{p}}\circ\mathrm{pr}_2^*\circ\varphi^*_{\mathfrak{p}}.$$

Here, the second step is because the morphism induced by isogeny commutes with base change. We note that for a point $(A,H)\in]\mathcal{M}^{\mathrm{tor,ord}}_{K,k_0}[^{\mathfrak{p}}_{\underline{r}'}$, the composite isogeny

$$A\xrightarrow{\check{\pi}_{\mathfrak{p}}} A/H\xrightarrow{\pi_{\mathfrak{p}}} (A/H)/C_{\mathfrak{p}}=A/A[\mathfrak{p}]=A\otimes_{\mathcal{O}_F}\mathfrak{p}^{-1}$$

is by definition the isogeny $[\varpi_{\mathfrak{p}}]$. Hence, we have $\check{\pi}_{\mathfrak{p}}^*\circ\pi_{\mathfrak{p}}^*=[\varpi_{\mathfrak{p}}]^*$, and

$$U_{\mathfrak{p}}\mathrm{Fr}_{\mathfrak{p}}=\mathrm{Tr}_{\mathrm{pr}_1}\circ[\varpi_{\mathfrak{p}}]^*\circ\mathrm{pr}_2^*\circ\varphi_{\mathfrak{p}}^*.$$

By (3.16.1), we have $\mathrm{pr}_2^*\circ\varphi_{\mathfrak{p}}^*=\mathrm{pr}_1^*\circ S_{\mathfrak{p}}^*$. It follows that

$$U_{\mathfrak{p}}\mathrm{Fr}_{\mathfrak{p}}=\mathrm{Tr}_{\mathrm{pr}_1}\circ[\varphi_{\mathfrak{p}}]^*\circ\mathrm{pr}_1^*\circ S_{\mathfrak{p}}^*=N_{F/\mathbb{Q}}(\mathfrak{p})[\varpi_{\mathfrak{p}}]^*S_{\mathfrak{p}}^*=N_{F/\mathbb{Q}}(\mathfrak{p})S_{\mathfrak{p}}.$$

□

For an analytic function g defined over a quasi-compact open subset $V\subset]X^{\mathrm{tor,ord}}_K[^{\mathfrak{p}}_{\underline{r}'}$, we define the norm

$$\|g\|_V:\ =\|(\mathrm{pr}_2^*)^{-1}g\|_{\mathrm{pr}_2(U)}.$$

Here, the norm $\|\bullet\|_{\mathrm{pr}_2(U)}$ is defined by using the integral model $\mathfrak{X}^{\mathrm{tor}}_K$.

Lemma 3.23. — *Let $U\subset]X^{\mathrm{tor,ord}}_K[$ be a quasi-compact admissible open subset, and g be a section of $\mathcal{O}_{\mathrm{pr}_1^{-1}(U)}$. We have*

$$\|\mathrm{Tr}_{\mathrm{pr}_1}(g)\|_U\le p^{-d_{\mathfrak{p}}}\|g\|_{\mathrm{pr}^{-1}(U)},$$

where $d_{\mathfrak{p}}=[F_{\mathfrak{p}}:\mathbb{Q}_p]$, and $\mathrm{Tr}_{\mathrm{pr}_1}:\Gamma(\mathrm{pr}_1^{-1}(U),\mathcal{O}_{\mathrm{pr}^{-1}(U)})\to\Gamma(U,\mathcal{O}_U)$ is the trace map.

Proof. — Since pr_2 is an isomorphism, one may write $g=\mathrm{pr}_2^*(h)$. Then, by definition, we have $\|g\|_{\mathrm{pr}^{-1}(U)}=\|h\|_{\mathrm{pr}_2(\mathrm{pr}_1^{-1}(U))}$. We may assume that $\|h\|=1$. Thus, h can be defined over the integral formal model $\mathfrak{X}^{\mathrm{tor,ord}}_K$. By $\mathrm{pr}_1=S_{\mathfrak{p}^{-1}}\circ\varphi_{\mathfrak{p}}\circ\mathrm{pr}_2$ (3.16.1), we have $\mathrm{Tr}_{\mathrm{pr}_1}=\mathrm{Tr}_{S_{\mathfrak{p}^{-1}}}\mathrm{Tr}_{\varphi_{\mathfrak{p}}}\mathrm{Tr}_{\mathrm{pr}_2}$. Note that $\mathrm{Tr}_{\mathrm{pr}_2}$ is the inverse of pr_2^*, since pr_2 is an isomorphism. Thus, we have $\mathrm{Tr}_{\mathrm{pr}_1}(g)=\mathrm{Tr}_{S_{\mathfrak{p}^{-1}}}(\mathrm{Tr}_{\varphi_{\mathfrak{p}}}(h))$. Since $S_{\mathfrak{p}^{-1}}$ is an automorphism of the integral model $\mathfrak{X}^{\mathrm{tor}}_K$, we have

$$\|\mathrm{Tr}_{S^{\mathfrak{p}^{-1}}}\circ\mathrm{Tr}_{\varphi_{\mathfrak{p}}}(h)\|_U=\|\mathrm{Tr}_{\varphi_{\mathfrak{p}}}(g)\|_{S_{\mathfrak{p}}(U)}.$$

It thus suffices to show that $\|\mathrm{Tr}_{\varphi_{\mathfrak{p}}}(g)\|_{S_{\mathfrak{p}}(U)}\le p^{-f_{\mathfrak{p}}}$. This follows from Lemma 3.14(2). □

Proposition 3.24. — *Let $U_1,U_2\subset]X^{\mathrm{tor,ord}}_K[$ be quasi-compact admissible open subsets in the ordinary locus such that $\mathrm{pr}_1^{-1}(U_1)\subset\mathrm{pr}_2^{-1}(U_2)$, and f be a section of $\underline{\omega}^{\epsilon_J(\underline{k},w)}\otimes\underline{\Omega}^J(-\mathsf{D})$ over U_2. We have*

$$\|U_{\mathfrak{p}}(f)\|_{U_1}\le p^{-(\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}\frac{w-k_\tau}{2}+\sum_{\tau\in(\Sigma_{\infty/\mathfrak{p}}-J)}(k_\tau-1))}\|f\|_{U_2}.$$

Proof. — Up to shrinking U_1 and U_2, we may assume that, for each $\tau\in\Sigma_\infty$, there exist

- a basis $(\omega_{\tau,i},\eta_{\tau,i})$ of $\mathcal{H}^1_\tau$ over U_2 adapted to the Hodge filtration $0\to\underline{\omega}_\tau\to\mathcal{H}^1_\tau\to\wedge^2(\mathcal{H}^1_\tau)\otimes\underline{\omega}_\tau^{-1}\to0$ and statisfying
$$\|\omega_{\tau,i}\|_{U_i}=\|\eta_{\tau,i}\|_{U_i}=1;\quad\text{and}$$
- a basis $dz_{J,i}$ of $\underline{\Omega}^J$ over U_i with $\|dz_{J,i}\|_{U_i}=1$.

We denote by $\overline{\eta}_{\tau,i}$ the image of $\eta_{\tau,i}$ in $\wedge^2(\mathcal{H}^1_\tau)\otimes\underline{\omega}_\tau^{-1}$. We write

$$f=g\Big(\bigotimes_{\tau\in\Sigma_\infty}(\omega_{\tau,2}\wedge\eta_{\tau,2})^{\frac{w-k_\tau}{2}}\bigotimes_{\tau\in J}\omega_{\tau,2}^{k_\tau-2}\otimes\bigotimes_{\tau\notin J}\overline{\eta}_{\tau,2}^{k_\tau-2}\Big)\otimes dz_{J,2},$$

where $g\in\Gamma(U_2,\mathcal{O}_{U_2})$. By definition, we have

$$U_{\mathfrak{p}}(f)=\mathrm{Tr}_{\mathrm{pr}_1}\Big(\mathrm{pr}_2^*(g|_{\mathrm{pr}_1^{-1}(U_1)})\check{\pi}_{\mathfrak{p}}^*\mathrm{pr}_2^*\Big(\bigotimes_{\tau\in\Sigma_\infty}(\omega_{\tau,2}\wedge\eta_{\tau,2})^{\frac{w-k_\tau}{2}}\bigotimes_{\tau\in J}\omega_{\tau,2}^{k_\tau-2}\bigotimes_{\tau\notin J}\overline{\eta}_{\tau,2}^{k_\tau-2}\Big)\otimes\mathrm{pr}_2^*(dz_{J,2})\Big).$$

There exist rigid analytic functions a_τ, b_τ on $\mathrm{pr}_1^{-1}(U_1)$ such that

$$\begin{cases} \breve{\pi}_{\mathfrak{p}}^* \mathrm{pr}_2^*(\omega_{\tau,2}) = a_\tau \mathrm{pr}_1^*(\omega_{\tau,1}), \\ \breve{\pi}_{\mathfrak{p}}^* \mathrm{pr}_2^*(\overline{\eta}_{\tau,2}) = b_\tau \mathrm{pr}_1^*(\overline{\eta}_{\tau,1}), \\ \breve{\pi}_{\mathfrak{p}}^* \mathrm{pr}_2^*(\omega_{\tau,2} \wedge \eta_{\tau,2}) = a_\tau b_\tau \mathrm{pr}_1^*(\omega_{\tau,1} \wedge \eta_{\tau,1}). \end{cases}$$

By Lemma 3.19, we have $\|a_\tau\|_{\mathrm{pr}_1^{-1}(U_1)} = 1$ for all $\tau \in \Sigma_\infty$, $\|b_\tau\|_{\mathrm{pr}_1^{-1}(U_1)} = 1$ for $\tau \notin \Sigma_{\infty/\mathfrak{p}}$ and $\|b_\tau\|_{\mathrm{pr}_1^{-1}(U_1)} = p^{-1}$ for $\tau \in \Sigma_{\infty/\mathfrak{p}}$. Similarly, there exists a rigid analytic function c_J such that $\mathrm{pr}_2^*(dz_{J,2}) = c_J \mathrm{pr}_1^*(dz_{J,1})$. By Lemma 3.18, we have $\|c_J\|_{\mathrm{pr}_1^{-1}(U_1)} = p^{|J \cap \Sigma_{k_0/\mathfrak{p}}|}$. So we obtain

$$U_{\mathfrak{p}}(f) = \mathrm{Tr}_{\mathrm{pr}_1}(\mathrm{pr}_2^*(g)h)(\otimes_{\tau \in \Sigma_\infty}(\omega_{\tau,1} \wedge \eta_{\tau,1})^{\frac{w-k_\tau}{2}} \otimes_{\tau \in J} \omega_{\tau,1}^{k_\tau - 2} \otimes_{\tau \notin J} \overline{\eta}_{\tau,1}^{k_\tau - 2}) \otimes dz_{J,1},$$

where $h = \prod_{\tau \in \Sigma_\infty}(a_\tau b_\tau)^{\frac{w-k_\tau}{2}} \prod_{\tau \in J} a_\tau^{k_\tau - 2} \prod_{\tau \notin J} b_\tau^{k_\tau - 2} c_J$. Now it follows from Lemma 3.23 that

$$\begin{aligned} \|U_{\mathfrak{p}}(f)\|_{U_1} &= \|\mathrm{Tr}_{\mathrm{pr}_1}\big(\mathrm{pr}_2^*(g)h\big)\|_{U_1} \\ &\leq p^{-d_{\mathfrak{p}} - \sum_{\tau \in \Sigma_{\infty/\mathfrak{p}}} \frac{w-k_\tau}{2} - \sum_{\tau \in (\Sigma_{\infty/\mathfrak{p}} - J)}(k_\tau - 2) + \#(J \cap \Sigma_{\infty/\mathfrak{p}})} \|g\|_{\mathrm{pr}_2(\mathrm{pr}^{-1}(U_1))} \\ &\leq p^{-\sum_{\tau \in \Sigma_{\infty/\mathfrak{p}}} \frac{w-k_\tau}{2} - \sum_{\tau \in (\Sigma_{\infty/\mathfrak{p}} - J)}(k_\tau - 1)} \|f\|_{U_2}. \end{aligned}$$

□

We deduce immediately from Proposition 3.24 the following

Corollary 3.25. — *Let $f \in S^\dagger_{\epsilon_J(\underline{k},w)}(K, L_\wp)$ be a generalized eigenform for $U_{\mathfrak{p}}$ with eigenvalue $\lambda_{\mathfrak{p}} \neq 0$. Then we have*

$$\mathrm{val}_p(\lambda_{\mathfrak{p}}) \geq \sum_{\tau \in \Sigma_{\infty/\mathfrak{p}}} \frac{w - k_\tau}{2} + \sum_{\tau \in (\Sigma_{\infty/\mathfrak{p}} - J)} (k_\tau - 1).$$

4. Formalism of Rigid Cohomology

In this section, we will relate the cohomology group $H^\star_{\mathrm{rig}}(X_K^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ to the rigid cohomology of the Goren-Oort strata of the Hilbert modular variety.

4.1. A brief recall of rigid cohomology. — We recall what we need on the rigid cohomology. For more details, we refer the reader to [**Be96**] and [**Ts99**]. Let $L_\wp$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_\wp$ the ring of integers and k_0 the residue field. Let $\mathcal{P}$ be a proper smooth formal scheme over $W(k_0)$, P its special fiber, and $\mathcal{P}_{\mathrm{rig}}$ the associated rigid analytic space. We have a natural specialization map $\mathrm{sp}\colon \mathcal{P}_{\mathrm{rig}} \to P$. For a locally closed subscheme $Z \subseteq P$, we put $]Z[_{\mathcal{P}} = \mathrm{sp}^{-1}(Z)$. When it is clear, we omit the subscript $\mathcal{P}$ from the notation.

For $\overline{X}$ be a locally closed subscheme of P, $j_X : X \to \overline{X}$ an open subset, and $\mathcal{E}$ a sheaf of abelian groups defined over some strict neighborhood of $]X[$ in $]\overline{X}[$, we put

$$j_X^\dagger \mathcal{E} = \varinjlim_V j_{V*} j_V^* \mathcal{E}$$

where V runs through a fundamental system of strict neighborhoods of $]X[$ inside $]\overline{X}[$ on which $\mathcal{E}$ is defined, and $j_V : V \to]\overline{X}[$ is the natural inclusion.

An *overconvergent F-isocrystal* $\mathscr{E}$ on $X/L_\wp$ can be viewed as a locally free coherent sheaf defined over some strict neighborhood V of $]X[$ inside $]\overline{X}[$, equipped with an integrable connection $\nabla : \mathscr{E} \to \mathscr{E} \otimes_{\mathcal{O}_V} \Omega^1_V$ satisfying certain (over)convergence conditions [**Be96**, Chap. 2], and, Zariski locally, an isomorphism $F^*\mathscr{E} \to \mathscr{E}$ where F is a Zariski local lift of the absolute Frobenius to $\mathcal{P}$. Let $\mathrm{DR}^\bullet(\mathscr{E}) = \mathscr{E} \otimes \Omega^\bullet_V$ be the associated de Rham complex. The *rigid cohomology* of $\mathscr{E}$ is defined to be

$$R\Gamma_{\mathrm{rig}}(X/L_\wp, \mathscr{E}) :\, = R\Gamma(]\overline{X}[, j^\dagger_X \mathrm{DR}^\bullet(\mathscr{E})).$$

When $\mathscr{E}$ is the constant F-isocrystal, we simply put $R\Gamma_{\mathrm{rig}}(X/L_\wp) = R\Gamma_{\mathrm{rig}}(X/L_\wp, \mathscr{E})$. For a sheaf $\mathcal{E}$ of abelian groups over a strict neighborhood of $]X[$, we define a sheaf on $]\overline{X}[$ by

$$\underline{\Gamma}_{]X[}(\mathcal{E}) :\, = \mathrm{Ker}(j^\dagger_X \mathcal{E} \to i_* i^* \mathcal{E}),$$

where $i :]\overline{X} - X[\to]\overline{X}[$ denotes the canonical immersion. Following Berthelot, the *rigid cohomology with compact support* of X with values in $\mathscr{E}$ is given by

$$R\Gamma_{c,\mathrm{rig}}(X/L_\wp, \mathscr{E}) :\, = R\Gamma(]\overline{X}[, \underline{\Gamma}_{]X[}(\mathrm{DR}^\bullet(\mathscr{E}))).$$

There is a natural map $R\Gamma_{c,\mathrm{rig}}(X, \mathscr{E}) \to R\Gamma_{\mathrm{rig}}(X, \mathscr{E})$ in the derived category, which induces maps on cohomology groups $H^\star_{c,\mathrm{rig}}(X, \mathscr{E}) \to H^\star_{\mathrm{rig}}(X/L_\wp, \mathscr{E})$.

Similarly, if Z is a closed subscheme of X, we define the functor $\underline{\Gamma}^\dagger_{]Z[}$ by

$$\underline{\Gamma}^\dagger_{]Z[}(\mathcal{E}) :\, = \mathrm{Ker}(j^\dagger_X \mathcal{E} \to j^\dagger_{X-Z} \mathcal{E})$$

for any sheaf of abelian groups $\mathcal{E}$ defined over a strict neighborhood of $]X[$ on $]\overline{X}[$. The functor $\underline{\Gamma}^\dagger_{]Z[}$ is exact. The *rigid cohomology with support in* Z of the F-isocrystal $\mathscr{E}$ is defined to be

$$R\Gamma_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E}) :\, = R\Gamma\big(]\overline{X}[, \underline{\Gamma}^\dagger_{]Z[}(\mathrm{DR}^\bullet(\mathscr{E}))\big).$$

There is a canonical distinguished triangle

$$R\Gamma_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E}|_Z) \to R\Gamma_{\mathrm{rig}}(X/L_\wp, \mathscr{E}) \to R\Gamma_{\mathrm{rig}}(X - Z/L_\wp, \mathscr{E}|_{X-Z}) \xrightarrow{+1}$$

In particular, one has canonical maps of cohomology groups

$$H^\star_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E}) \to H^\star_{\mathrm{rig}}(X/L_\wp, \mathscr{E}). \tag{4.1.1}$$

If Z is closed in $\overline{X}$ (equivalently, Z is proper over k_0), then this map factor through $H^\star_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E}) \to H^\star_{c,\mathrm{rig}}(X/L_\wp, \mathscr{E})$. It is standard that $H^\star_{\mathrm{rig}}(X/L_\wp, \mathscr{E})$ and $H^\star_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E})$ are independent of the embedding $X \hookrightarrow P$ and the choice of formal model $\mathcal{P}$. We remark that if U is an open subscheme of X containing Z, then we have a natural isomorphism $H^*_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E}) \cong H^*_{Z,\mathrm{rig}}(U/L_\wp, \mathscr{E}|_U)$ [**Ts99**, Proposition 2.1.1].

When X is smooth of pure dimension d_X, then there exists a perfect Poincaré duality between $H^\star_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E})$ and $H^{2d_X-\star}_{c,\mathrm{rig}}(Z/L_\wp, \mathscr{E}^\vee)$ [**Ke06**, Theorem 1.2.3]. Let d_Z be the dimension of Z and put $c = d_X - d_Z$. We denote by $c(Z) \in H^{2c}_{Z,\mathrm{rig}}(X/L_\wp)$ the cohomolgy class via Poincaré duality corresponding to the trace map $\mathrm{Tr}_Z : H^{2d_Z}_{c,\mathrm{rig}}(Z/L_\wp) \to L_\wp$. We call it the *cycle class* of Z in X.

If Z is also smooth, then the cup product by $c(Z)$ defines a natural Gysin isomorphism

$$H^\star_{\mathrm{rig}}(Z/L_\wp, \mathscr{E}|_Z) \xrightarrow{\cong} H^{\star+2c}_{Z,\mathrm{rig}}(X/L_\wp, \mathscr{E}). \tag{4.1.2}$$

Actually, this is proved in [**Ts99**, Theorem 4.1.1] under the additional assumption that X and Z are both affine, and can be lifted to smooth admissible formal schemes $\mathfrak{X}$ and $\mathfrak{Z}$ over $W(k_0)$ and such that $\mathfrak{Z}$ is globally the intersection of d normal crossing smooth divisors of $\mathfrak{X}$. To prove the statement in the general case, one can proceed in two ways: either one covers X by open affine subsets $\{U_\alpha : \alpha \in I\}$ such that each $(Z \cap U_\alpha, U_\alpha)$ satisfies the additional assumption above, then one computes the both sides of (4.1.2) with certain Čech complex with respect to $\{U_\alpha : \alpha \in I\}$; or one can use the Poincaré duality proved in [**Ke06**, Theorem 1.2.3] to identify the both sides of (4.1.2) with the dual of $H^{2d_Z-\star}_{c,\mathrm{rig}}(Z/L_\wp, \mathscr{E}^\vee)$, the rigid cohomology with compact support of the dual isocrystal $\mathscr{E}^\vee$.

Combining (4.1.1) and (4.1.2), we get a Gysin map

$$\mathrm{G}_{\mathrm{Z},\mathscr{E}} \colon H^\star_{\mathrm{rig}}(Z/L_\wp, \mathscr{E}|_Z) \to H^{\star+2d}_{\mathrm{rig}}(X/L_\wp, \mathscr{E})(d).$$

If Z is proper over k_0, this map factors through the natural morphism $H^{\star+2d}_{c,\mathrm{rig}}(X/L_\wp,\mathscr{E})(d)\to H^{\star+2d}_{\mathrm{rig}}(X/L_\wp,\mathscr{E})(d)$.

4.2. Formalism of dual Čech complex. — Let Σ denote a finite set. Assume that, to each subset $\mathtt{T}\subseteq\Sigma$, there is an associated $\mathbb{Q}$-vector space $M_{\mathtt{T}}$ such that for each inclusion of subsets $\mathtt{T}_1\subseteq\mathtt{T}_2$, we have an (ordering reversing) $\mathbb{Q}$-linear map $i_{\mathtt{T}_2,\mathtt{T}_1}: M_{\mathtt{T}_2}\to M_{\mathtt{T}_1}$ satisfying the natural cocycle condition. We consider some formal symbols e_τ, called the *Čech symbols*, indexed by elements $\tau\in\Sigma$, and their formal wedge products in the sense that $e_\tau\wedge e_{\tau'}=-e_{\tau'}\wedge e_\tau$ for $\tau,\tau'\in\Sigma$. For a subset $\mathtt{T}=\{\tau_1,\dots,\tau_i\}$ of Σ, we fix an order for it and write $e_{\mathtt{T}}$ for $e_{\tau_1}\wedge\cdots\wedge e_{\tau_i}$. The *dual Čech complex* associated to $M_{\mathtt{T}}$ is then given by

$$M_\Sigma e_\Sigma\to\cdots\to\bigoplus_{\#\mathtt{T}=1}M_{\mathtt{T}}e_{\mathtt{T}}\to\bigoplus_{\#\mathtt{T}=2}M_{\mathtt{T}}e_{\mathtt{T}}\to\cdots\to M_\emptyset,$$

where the connecting homomorphism is given by, for $\mathtt{T}=\{\tau_1,\dots,\tau_i\}$,

$$m_{\mathtt{T}}e_{\tau_1}\wedge\cdots\wedge e_{\tau_i}\mapsto\sum_{j=1}^{i}(-1)^j i_{\mathtt{T},\mathtt{T}-\{\tau_j\}}(m_{\mathtt{T}})e_{\tau_1}\wedge\cdots e_{\tau_{j-1}}\wedge e_{\tau_{j+1}}\wedge\cdots\wedge e_{\tau_i}.$$

It is clear from the construction that this is a complex. Note that when $M_{\mathtt{T}}=M$ for all $\mathtt{T}\subseteq\Sigma$ and $i_{\mathtt{T}_2,\mathtt{T}_1}=\mathrm{Id}_M$ for all $\mathtt{T}_1\subset\mathtt{T}_2$, the dual Čech complex associated to $M_{\mathtt{T}}$ is acyclic.

Lemma 4.3. — *Let the notation be as in Subsection 4.1. Let $Y=\bigcup_{\tau\in\Sigma}Y_\Sigma$ be a finite union of closed subschemas of X. For any subset $\mathtt{T}\subseteq\Sigma$, we put $Y_{\mathtt{T}}=\bigcap_{\tau\in\mathtt{T}}Y_\tau$, and let $j_{\mathtt{T}}: X-Y_{\mathtt{T}}\to\overline{X}$ denote the natural immersion. For any sheaf $\mathcal{E}$ of abelian groups defined on a strict neighborhood of $]\overline{X}-Y[$, the sequence*

$$0\to j^\dagger_\Sigma\mathcal{E}\ e_\Sigma\to\bigoplus_{\tau\in\Sigma}j^\dagger_{\Sigma\backslash\{\tau\}}\mathcal{E}\ e_{\Sigma\backslash\{\tau\}}\to\cdots\to\bigoplus_{\tau\in\Sigma}j^\dagger_{\{\tau\}}\mathcal{E}\ e_\tau\to j^\dagger_{\overline{X}-Y}\mathcal{E}\ e_\emptyset\to 0. \tag{4.3.1}$$

given with dual Čech complex is exact. Here, we place $j^\dagger_{\overline{X}-Y}\mathcal{E}\ e_\emptyset$ at degree 0, and the $(-i)$-th term is a direct sum, over all subsets $\mathtt{T}\subseteq\Sigma$ with $\#\mathtt{T}=i$, of $j^\dagger_{\mathtt{T}}\mathcal{E}\ e_{\mathtt{T}}$, and all the morphisms are natural restriction maps.

Proof. — We prove the Lemma by induction on $\#\Sigma$. When $\#\Sigma=1$, the statement is trivial. Assume now that the Lemma holds for $\#\Sigma=n-1$, and we need to prove it for $\#\Sigma=n$. For each $\tau\in\Sigma$, let V_τ be a strict neighborhood of $]\overline{X}-Y_\tau[$. Then $(\bigcup_{\tau\in\Sigma}V_\tau)\cup]Y_\Sigma[$ form an admissible covering of $]\overline{X}[$. The restriction of the sequence in question to $]Y_\Sigma[$ is identically zero, it suffices to prove its exactness when restricted to each V_τ. By standard arguments of direct limits, it is enough to prove the exactness of (4.3.1) after applying $j^\dagger_{\{\tau\}}$. Note that $j^\dagger_{\{\tau\}}j^\dagger_{\mathtt{T}}=j^\dagger_{\{\tau\}}$ if $\tau\in\mathtt{T}$, and $j^\dagger_{\{\tau\}}j^\dagger_{\overline{X}-Y}=j^\dagger_{\overline{X}-Y}$. It is easy to see that after applying $j^\dagger_{\{\tau\}}$, the resulting complex is the direct sum of a complex of type (4.3.1) but with X replaced by $X'=X-Y_\tau$ and Y replaced by $Y'=\bigcup_{\tau'\in\Sigma\backslash\{\tau\}}Y_{\tau'}$, and the dual Čech complex concentrated in degrees $[-n,-1]$ with constant group $j^\dagger_{\{\tau\}}\mathcal{E}$. By the last remark of the previous Subsection, the latter is acyclic. Hence, the desired exactness follows from the induction hypothesis. □

4.4. Setup of Hilbert modular varieties. — Let L, $L_\wp$, $\mathcal{O}_\wp$ and k_0 be as in Subsection 3.1. We fix an open subgroup $K=K_pK^p$ such that $K_p=\mathrm{GL}_2(\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p)$, and K^p satisfies Hypothesis 2.7. To simplify notation, let $\mathbf{X}$ denote the base change to $W(k_0)$ of the integral model of Shimura variety $\mathbf{Sh}_K(G)$ considered in Subsection 2.3. Let $\mathbf{X}^{\mathrm{tor}}$ be a toroidal compactification of $\mathbf{X}$ as in Subsection 2.10. We use X and X^{tor} to denote their special fibers over k_0. Let $\mathfrak{X}^{\mathrm{tor}}$ be the formal completion of $\mathbf{X}$ along its special fiber, and $\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}$ denote the base change to $L_\wp$ of the rigid analytic spaces associated to $\mathbf{X}^{\mathrm{tor}}$. Let $\mathfrak{X}\subset\mathfrak{X}^{\mathrm{tor}}$ denote the open formal subscheme corresponding to X. For a sub variety $Z\subseteq X^{\mathrm{tor}}$, we denote by $]Z[=]Z[_{\mathfrak{X}^{\mathrm{tor}}}$ the tube of Z in $\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}$.

For $\tau \in \Sigma_\infty$, let Y_τ denote the vanishing locus of the partial Hasse invariant h_τ at $\tau \in \Sigma_\infty$ defined in Subsection 3.2. Note that Y_τ has no intersection with the toroidal boundary D. We put $Y = \bigcup_{\tau\in\Sigma_\infty} Y_\tau$, and $X^{\mathrm{tor,ord}} = X^{\mathrm{tor}} - Y$ and $X^{\mathrm{ord}} = X^{\mathrm{tor,ord}} \cap X$. For a subset $\mathtt{T} \subseteq \Sigma_\infty$, we put $Y_{\mathtt{T}} = \cap_{\tau\in\mathtt{T}} Y_\tau$. Then it is a smooth closed sub-variety of X^{tor} of codimension $\#\mathtt{T}$, and we call it a *closed Goren-Oort stratum* (or GO-stratum for short) of codimension $\#\mathtt{T}$. By convention, we also put $Y_\emptyset = X$.

4.5. Isocrystals on the Hilbert modular varieties. — Let $\mathcal{A}^{\mathrm{sa}}$ denote the family of semi-abelian varieties over X^{tor} which extends the universal HBAV $\mathcal{A}$ on X. Let $(X/W(k_0))_{\mathrm{cris}}$ denote the crystalline site of X relative to the natural divided power structure on $(p) \subset W(k_0)$. Then the relative crystalline cohomology $\mathcal{H}^1_{\mathrm{cris}}(\mathcal{A}/X)$ is an F-crystal over $(X/W(k_0))_{\mathrm{cris}}$. The evaluation of $\mathcal{H}^1_{\mathrm{cris}}(\mathcal{A}/X)$ at the divided power embedding $X \to \mathfrak{X}$ is canonically identified with the relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{X})$, where $\mathcal{A}$ also denotes the universal HBAV over $\mathfrak{X}$ by abuse of notation. We denote by $\mathscr{D}(\mathcal{A})$ the (overconvergent) F-isocrystal on $X/W(k_0)[1/p]$ (hence also an isocrytsal over $X/L_\wp$ by base change) associated to $\mathcal{H}^1_{\mathrm{cris}}(\mathcal{A}/X)$. The action of $\mathcal{O}_F$ on $\mathcal{A}$ induces an action of $\mathcal{O}_F$ on $\mathscr{D}(\mathcal{A})$, and we have a natural decomposition

$$\mathscr{D}(\mathcal{A}) = \oplus_{\tau\in\Sigma_\infty} \mathscr{D}(\mathcal{A})_\tau,$$

where each $\mathscr{D}(\mathcal{A})_\tau$ is a log-isocrystal of rank 2.

For a multiweight $(\underline{k}, w)$, we put

$$\mathscr{D}^{(\underline{k},w)} : = \bigotimes_{\tau\in\Sigma_\infty} (\wedge^2 \mathscr{D}(\mathcal{A})_\tau)^{\frac{w-k_\tau}{2}} \otimes \mathrm{Sym}^{k_\tau-2} \mathscr{D}(\mathcal{A})_\tau.$$

This is an F-isocrystal over $X/L_\wp$, and its evaluation on $\mathfrak{X}$ is the vector bundle $\mathscr{F}^{(\underline{k},w)}$ defined in Subsection 2.12 on the rigid analytic variety $\mathfrak{X}_{\mathrm{rig}}$. Note that $\mathscr{F}^{(\underline{k},w)}$ extends to a vector bundle over $\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}$ equipped with an integrable connection with logarithmic poles along D (Subsection 2.12). For each sub-variety $Z \subset X$, the rigid cohomology of Z with values in $\mathscr{D}^{(\underline{k},w)}$) can be computed as

$$H^\star_{\mathrm{rig}}(Z/L_\wp, \mathscr{D}^{(\underline{k},w)}) = H^\star(]Z[, j_Z^\dagger \mathrm{DR}^\bullet(\mathscr{F}^{(\underline{k},w)})),$$

where j_Z denote the canonical inclusion $]Z[\hookrightarrow \mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}$.

4.6. Partial Frobenius on X. — Let S be a connected locally noetherian $\mathbb{F}_p$-scheme, and $x = (A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$ be an S-valued point of X. For each $\mathfrak{p} \in \Sigma_p$, we construct a new point $\varphi_{\mathfrak{p}}(x) = (A', \iota', \bar{\lambda}', \bar{\alpha}'_{K^p})$ of X as follows:

- Let $\mathrm{Ker}_{\mathfrak{p}}$ denote the $\mathfrak{p}$-component of the kernel of the relative Frobenius homomorphism $\mathrm{Fr}_A : A \to A^{(p)}$. We put $A' = A/\mathrm{Ker}_{\mathfrak{p}}$, and equip it with the induced action ι' of $\mathcal{O}_F$. Let $\pi_{\mathfrak{p}} : A \to A'$ denote the canonical isogeny.
- If λ is a $\mathfrak{c}$-polarization on A, then it induces a natural $\mathfrak{cp}$-polarization on A' determined by the commutative diagram:

$$\begin{array}{ccc} A' \otimes_{\mathcal{O}_F} \mathfrak{cp} & \xrightarrow{\check{\pi}_{\mathfrak{p}}} & A \otimes_{\mathcal{O}_F} \mathfrak{c} \\ \lambda' \big\downarrow \cong & & \lambda \big\downarrow \cong \\ A'^\vee & \xrightarrow{\pi_{\mathfrak{p}}^\vee} & A^\vee. \end{array}$$

  Here, $\check{\pi}_{\mathfrak{p}}$ is the unique map such that the composite $A \otimes_{\mathcal{O}_F} \mathfrak{cp} \xrightarrow{\pi_{\mathfrak{p}}} A' \otimes_{\mathcal{O}_F} \mathfrak{cp} \xrightarrow{\check{\pi}_{\mathfrak{p}}} A \otimes_{\mathcal{O}_F} \mathfrak{c}$ is the canonical quotient map by $A[\mathfrak{p}]$.
- The K^p-level structure α'_{K^p} on A' is the unique one induced by isomorphism $\pi_{\mathfrak{p},*} : T^{(p)}(A) \xrightarrow{\sim} T^{(p)}(A')$ of prime-to-p Tate modules.

With the convention in Remark 2.8, $(A', \iota', \bar{\lambda}', \bar{\alpha}_{K^p})$ well defines a point on X. We denote by $\varphi_{\mathfrak{p}} : X \to X$ the obtained endomorphism of the Hilbert modular variety. It is finite and flat of degree $p^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}$. By choosing appropriate cone decompositions, one may assume that $\varphi_{\mathfrak{p}}$ extends to an endomorphism of X^{tor}. It is clear that the restriction of $\varphi_{\mathfrak{p}}$ to the ordinary locus $X^{\mathrm{tor,ord}}$ coincides with the reduction of $\varphi_{\mathfrak{p}} : \mathfrak{X}^{\mathrm{tor,ord}} \to \mathfrak{X}^{\mathrm{tor,ord}}$ considered in Subsection 3.13, since the $\mathfrak{p}$-canonical subgroups there lift $\mathrm{Ker}_{\mathfrak{p}}$. Note that $\varphi_{\mathfrak{p}}$ and $\varphi_{\mathfrak{q}}$ with $\mathfrak{p} \neq \mathfrak{q}$ commute with each other, and the product $F_{X/\mathbb{F}_p} = \prod_{\mathfrak{p}\in\Sigma_p} \varphi_{\mathfrak{p}} : X \to X$ is the Frobenius endomorphism of X relative to $\mathbb{F}_p$. We call $\varphi_{\mathfrak{p}}$ the $\mathfrak{p}$-*partial Frobenius*. Let $\sigma_{\mathfrak{p}} : \Sigma_\infty \to \Sigma_\infty$ be the map defined by

$$\sigma_{\mathfrak{p}}(\tau) = \begin{cases} \tau & \text{if } \tau \notin \Sigma_{\infty/\mathfrak{p}}, \\ \sigma\tau & \text{if } \tau \in \Sigma_{\infty/\mathfrak{p}}. \end{cases}$$

For an subset $\mathtt{T} \subseteq \Sigma_\infty$, we denote by $\sigma_{\mathfrak{p}}\mathtt{T}$ its image under $\sigma_{\mathfrak{p}}$.

Lemma 4.7. — *Let $x = (A, \iota, \bar{\lambda}, \bar{\alpha}_{K^p})$ be a point of X with values in a k_0-scheme S, and $\varphi_{\mathfrak{p}}(x) = (A', \iota', \bar{\lambda}', \bar{\alpha}'_{K^p})$. Then τ-th partial Hasse invariant $h_\tau(\varphi_{\mathfrak{p}}(x))$ is canonically identified with $h_\tau(x)$ if $\tau \notin \Sigma_{\infty/\mathfrak{p}}$, and with $h_{\sigma^{-1}\tau}(x)^{\otimes p}$ if $\tau \in \Sigma_{\infty/\mathfrak{p}}$; in particular, if S is the spectrum of a perfect field, then $h_\tau(\varphi_{\mathfrak{p}}(x)) = 0$ if and only if $h_{\sigma_{\mathfrak{p}}^{-1}\tau}(x) = 0$.*

Proof. — The statement is clear for $\tau \notin \Sigma_{\infty/\mathfrak{p}}$. Now suppose that $\tau \in \Sigma_{\infty/\mathfrak{p}}$. As $\mathfrak{p}$ is unramified, $A'[\mathfrak{p}^\infty]$ is the quotient of $A[\mathfrak{p}^\infty]$ by its kernel of Frobenius, hence there exists an isomorphism of p-divisible groups $A'[\mathfrak{p}^\infty] \simeq (A[\mathfrak{p}^\infty])^{(p)}$. There exists thus an isomorphism

$$\omega_{A'/S,\tau} = \omega_{A'[\mathfrak{p}^\infty]/S,\tau} \cong \omega^{(p)}_{A/S,\sigma^{-1}\tau}$$

compatible with the morphism induced by the Verschiebung. It follows that $h_\tau(A')$ is identified with the base change of $h_{\sigma^{-1}\tau}(A)$ via the absolute Frobenius on S, whence the Lemma. □

Corollary 4.8. — *For a subset $\mathtt{T} \subseteq \Sigma_\infty$, the restriction of the partial Frobenius $\varphi_{\mathfrak{p}}$ to $Y_{\mathtt{T}}$ defines a finite flat map $\varphi_{\mathfrak{p}} : Y_{\mathtt{T}} \to Y_{\sigma_{\mathfrak{p}}\mathtt{T}}$ of degree $p^{\#(\Sigma_{\infty/\mathfrak{p}} - \mathtt{T}_{\mathfrak{p}})}$, with $\mathtt{T}_{\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}} \cap \mathtt{T}$. If $\varphi_{\mathfrak{p}}^{-1}(Y_{\sigma_{\mathfrak{p}}\mathtt{T}})$ be the fiber product of $\varphi_{\mathfrak{p}} : X \to X$ with the closed immersion $Y_{\sigma_{\mathfrak{p}}\mathtt{T}} \hookrightarrow X$, then we have an equality $[\varphi_{\mathfrak{p}}^{-1}(Y_{\sigma_{\mathfrak{p}}\mathtt{T}})] = p^{\#\mathtt{T}_{\mathfrak{p}}}[Y_{\mathtt{T}}]$ in the group algebraic cycles on X of codimension $\#\mathtt{T}$.*

Proof. — The Lemma implies that $\varphi_{\mathfrak{p}}$ sends $Y_{\mathtt{T}}$ to $Y_{\sigma_{\mathfrak{p}}\mathtt{T}}$. We note that $\prod_{\mathfrak{p}\in\Sigma_p} \varphi_{\mathfrak{p}} : Y_{\mathtt{T}} \to Y_{\mathtt{T}}^{(p)}$ is the relative Frobenius of $Y_{\mathtt{T}}$, which is finite flat. The flatness criterion by fibres implies the finite flatness of $\varphi_{\mathfrak{p}}|_{Y_{\mathtt{T}}}$. By the Lemma, $\varphi_{\mathfrak{p}}^{-1}(Y_{\sigma_{\mathfrak{p}}\mathtt{T}})$ is the closed subscheme of X defined by vanishing of h_τ's for $\tau \in \mathtt{T} - \mathtt{T}_{\mathfrak{p}}$ and $h_\tau^{\otimes p}$'s for $\tau \in \mathtt{T}_{\mathfrak{p}}$. Hence, $Y_{\mathtt{T}}$ is the closed subscheme of $\varphi_{\mathfrak{p}}^{-1}(Y_{\sigma_{\mathfrak{p}}\mathtt{T}})$ defined by the vanishing of h_τ's for $\tau \in \mathtt{T}_{\mathfrak{p}}$. Since $Y_{\mathtt{T}}$ is smooth, the equality $[\varphi_{\mathfrak{p}}^{-1} Y_{\sigma_{\mathfrak{p}}\mathtt{T}}] = p^{\#\mathtt{T}_{\mathfrak{p}}}[Y_{\mathtt{T}}]$ follows immediately. Note that $\varphi_{\mathfrak{p}}^{-1}(Y_{\sigma_{\mathfrak{p}}\mathtt{T}})$ is finite flat of degree $p^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}$ over $Y_{\sigma_{\mathfrak{p}}\mathtt{T}}$. Hence, the flat map $\varphi_{\mathfrak{p}}|_{Y_{\mathtt{T}}}$ must have degree $p^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}/p^{\#\mathtt{T}_{\mathfrak{p}}} = p^{\#(\Sigma_{\infty/\mathfrak{p}} - \mathtt{T}_{\mathfrak{p}})}$. □

We have the isogeny $\pi_{\mathfrak{p}} : \mathcal{A}^{\mathrm{sa}} \to \varphi_{\mathfrak{p}}^* \mathcal{A}^{\mathrm{sa}}$ by quotienting the subgroup $\mathrm{Ker}_{\mathfrak{p}}$ of $\mathcal{A}^{\mathrm{sa}}$. This induces an isomorphism of F-isocrystals:

$$\pi_{\mathfrak{p}}^* : \varphi_{\mathfrak{p}}^* \mathscr{D}(\mathcal{A}) = \mathscr{D}(\varphi_{\mathfrak{p}}^* \mathcal{A}) \xrightarrow{\simeq} \mathscr{D}(\mathcal{A}),$$

and hence an isomorphism $\pi_{\mathfrak{p}}^* : \varphi_{\mathfrak{p}}^* \mathscr{D}^{(\underline{k},w)} \simeq \mathscr{D}^{(\underline{k},w)}$. This gives rise to an operator $\mathrm{Fr}_{\mathfrak{p}}$ on the rigid cohomology for each $Y_{\mathtt{T}}$ with $\mathtt{T} \subseteq \Sigma_\infty$:

$$\mathrm{Fr}_{\mathfrak{p}} \colon H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}) \xrightarrow{\varphi_{\mathfrak{p}}^*} H^\star_{c,\mathrm{rig}}(Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T}}/L_\wp, \varphi_{\mathfrak{p}}^* \mathscr{D}^{(\underline{k},w)}|_{Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T}}}) \xrightarrow{\pi_{\mathfrak{p}}^*} H^\star_{c,\mathrm{rig}}(Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T}}}).$$

Here, we put $Y_\emptyset$ means X by convention, and $H^{j-2i}_{c,\mathrm{rig}}\big(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}\big)$ is the same as the usual rigid cohomology without compact support if $\mathtt{T} \neq \emptyset$. Similarly, we have an operator $\mathrm{Fr}_\mathfrak{p}$ on $H^{\star+2\#\mathtt{T}}_{Y_{\mathtt{T}},\mathrm{rig}}(X, \mathscr{D}^{(\underline{k},w)})$ such that the following diagram is commutative:

(4.8.1)
$$\begin{array}{ccc} H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}) & \xrightarrow[\text{Gysin}]{\simeq} & H^{\star+2\#\mathtt{T}}_{c,Y_{\mathtt{T}}}(X/L_\wp, \mathscr{D}^{(\underline{k},w)}) \\ \Big\downarrow {\scriptstyle p^{\#\mathtt{T}_\mathfrak{p}}\mathrm{Fr}_\mathfrak{p}} & & \Big\downarrow {\scriptstyle \mathrm{Fr}_\mathfrak{p}} \\ H^\star_{c,\mathrm{rig}}(Y_{\sigma_\mathfrak{p}^{-1}\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\sigma_\mathfrak{p}^{-1}\mathtt{T}}}) & \xrightarrow[\text{Gysin}]{\simeq} & H^{\star+2\#\mathtt{T}}_{Y_{\sigma_\mathfrak{p}^{-1}\mathtt{T}}}(X/L_\wp, \mathscr{D}^{(\underline{k},w)}). \end{array}$$

Here, $p^{\#\mathtt{T}_\mathfrak{p}}$ appears on the left vertical arrow, because the inverse image of cycle class $c(Y_{\mathtt{T}}) \in H^{2\#\mathtt{T}}_{Y_{\mathtt{T}},\mathrm{rig}}(X/L_\wp)$ under $\varphi_\mathfrak{p}^*$ is the class

$$\varphi_\mathfrak{p}^* c(Y_{\mathtt{T}}) = c(\varphi_\mathfrak{p}^{-1}(Y_{\mathtt{T}})) = p^{\#\mathtt{T}_\mathfrak{p}} c(Y_{\sigma_\mathfrak{p}^{-1}\mathtt{T}}) \in H^{2\#\mathtt{T}}_{Y_{\sigma_\mathfrak{p}^{-1}\mathtt{T}},\mathrm{rig}}(X/L_\wp),$$

where we used Corollary 4.8 and basic properties of class cycle map [**Pe03**].

Recall that we have, for each $\mathfrak{p} \in \Sigma_p$, an automorphism $S_\mathfrak{p}$ on X^{tor} defined in Subsection 3.10. We have an natural isogeny

$$[\varpi_\mathfrak{p}] : \mathcal{A} \to S_\mathfrak{p}^*\mathcal{A} = \mathcal{A} \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1}$$

which induces an isomorphism of isocrystals $[\varpi_\mathfrak{p}]^* : S_\mathfrak{p}^*\mathscr{D}^{(\underline{k},w)} \simeq \mathscr{D}^{(\underline{k},w)}$. Since $Y_{\mathtt{T}}$ is stable under $S_\mathfrak{p}$ for each $\mathtt{T} \subseteq \Sigma_\infty$, $S_\mathfrak{p}$ induces an automorphism

$$S_\mathfrak{p} : H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}) \to H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}).$$

4.9. Twisted partial Frobenius. — In order to compare with the unitary setting later, we define the *twisted partial Frobenius* to be

$$\mathfrak{F}_{\mathfrak{p}^2} := \varphi_\mathfrak{p}^2 \circ S_\mathfrak{p}^{-1} : X^{\mathrm{tor}} \to X^{\mathrm{tor}}.$$

Note that $\mathfrak{F}_{\mathfrak{p}^2}$ sends a point $(A, \iota, \bar\lambda, \bar\alpha_{K^p})$ to $((A/\operatorname{Ker}_{\mathfrak{p}^2}) \otimes_{\mathcal{O}_F} \mathfrak{p}, \iota', \bar\lambda', \bar\alpha'_{K^p})$, where $\operatorname{Ker}_{\mathfrak{p}^2}$ is the $\mathfrak{p}$-component of the kernel of the relative p^2-Frobenius $A \to A^{(p^2)}$. It is clear that $\mathfrak{F}_{\mathfrak{p}^2}$ send a GO-stratum $Y_{\mathtt{T}}$ to $Y_{\sigma_\mathfrak{p}^2\mathtt{T}}$. We use $\eta : \mathcal{A} \to \mathfrak{F}_{\mathfrak{p}^2}^*\mathcal{A}$ to denote the canonical quasi-isogeny

$$\mathcal{A} \to \mathcal{A}/\operatorname{Ker}_{\mathfrak{p}^2} \leftarrow \mathfrak{F}_{\mathfrak{p}^2}^*\mathcal{A} = (\mathcal{A}/\operatorname{Ker}_{\mathfrak{p}^2}) \otimes_{\mathcal{O}_F} \mathfrak{p}.$$

It induces an isomorphism of F-isocrytals $\eta_\mathfrak{p}^* : \mathfrak{F}_{\mathfrak{p}^2}^*\mathscr{D}(\mathcal{A}) = \mathscr{D}(\mathfrak{F}_{\mathfrak{p}^2}^*\mathcal{A}) \xrightarrow{\simeq} \mathscr{D}(\mathcal{A})$, and hence an isomorphism $\eta_\mathfrak{p}^* : \mathfrak{F}_{\mathfrak{p}^2}^*\mathscr{D}^{(\underline{k},w)} \xrightarrow{\simeq} \mathscr{D}^{(\underline{k},w)}$. For $\mathtt{T} \subseteq \Sigma_\infty$ we define the operator $\Phi_{\mathfrak{p}^2}$ on the rigid cohomology to be

(4.9.1)
$$\Phi_{\mathfrak{p}^2} : H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}) \xrightarrow{\mathfrak{F}_{\mathfrak{p}^2}^*} H^\star_{c,\mathrm{rig}}(Y_{\sigma_\mathfrak{p}^{-2}\mathtt{T}}/L_\wp, \mathfrak{F}_{\mathfrak{p}^2}^*\mathscr{D}^{(\underline{k},w)}|_{Y_{\sigma_\mathfrak{p}^{-2}\mathtt{T}}}) \xrightarrow{\eta_\mathfrak{p}^*} H^\star_{c,\mathrm{rig}}(Y_{\sigma_\mathfrak{p}^{-2}\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\sigma_\mathfrak{p}^{-2}\mathtt{T}}}).$$

It is clear that $\Phi_{\mathfrak{p}^2} = \mathrm{Fr}_\mathfrak{p}^2 \cdot S_\mathfrak{p}^{-1}$, and $\Phi_{\mathfrak{p}^2}$ commute with $\Phi_{\mathfrak{q}^2}$ for $\mathfrak{p}, \mathfrak{q} \in \Sigma_p$. Similarly to the case for $\mathrm{Fr}_\mathfrak{p}$, we also have an operator $\Phi_{\mathfrak{p}^2}$ on $H^{\star+2\#\mathtt{T}}_{Y_{\mathtt{T}},\mathrm{rig}}(X, \mathscr{D}^{(\underline{k},w)})$ such that the following diagram is commutative:

(4.9.2)
$$\begin{array}{ccc} H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}) & \xrightarrow[\text{Gysin}]{\simeq} & H^{\star+2\#\mathtt{T}}_{Y_{\mathtt{T}}}(X/L_\wp, \mathscr{D}^{(\underline{k},w)}) \\ \Big\downarrow {\scriptstyle p^{2\#\mathtt{T}_\mathfrak{p}}\Phi_{\mathfrak{p}^2}} & & \Big\downarrow {\scriptstyle \Phi_{\mathfrak{p}^2}} \\ H^\star_{c,\mathrm{rig}}(Y_{\sigma_\mathfrak{p}^{-2}\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}|_{Y_{\sigma_\mathfrak{p}^{-2}\mathtt{T}}}) & \xrightarrow[\text{Gysin}]{\simeq} & H^{\star+2\#\mathtt{T}}_{Y_{\sigma_\mathfrak{p}^{-2}\mathtt{T}}}(X/L_\wp, \mathscr{D}^{(\underline{k},w)}). \end{array}$$

Recall that we have defined the cohomology group $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F})$ in Subsection 3.4. Its relation with the rigid cohomology of the strata $Y_{\mathtt{T}}$ is given by

Proposition 4.10. — (1) *There exists a spectral sequence in the second quadrant*

$$E_1^{-i,j}=\bigoplus_{\#\mathtt{T}=i}H^{j-2i}_{c,\mathrm{rig}}\big(Y_{\mathtt{T}}/L_\wp,\mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}\big)e_{\mathtt{T}}\Rightarrow H^{j-i}_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)}). \tag{4.10.1}$$

Here, the $e_{\mathtt{T}}$*'s are the Čech symbols from Subsection 4.2, and the transition maps* $d_1^{-i,j}\colon E_1^{-i,j}\to E_1^{1-i,j}$ *are direct sums of the Gysin maps* $H^{j-2i}_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L,\mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}})\to H^{j-2i+2}_{c,\mathrm{rig}}(Y_{\mathtt{T}'}/L,\mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}'}})$ *with* $\mathtt{T}'\subseteq\mathtt{T}$ *and* $\#\mathtt{T}'=\#\mathtt{T}-1=i-1$ *using the dual Čech complex formalism in Subsection 4.2.*

(2) *The spectral sequence is equivariant under the natural action of tame Hecke algebra* $\mathscr{H}(K^p,L_\wp)=L_\wp[K^p\backslash\mathrm{GL}_2(\mathbb{A}^{\infty,p})/K^p]$*, and the actions of* $\mathrm{Fr}_{\mathfrak{p}}$ *for each* $\mathfrak{p}\in\Sigma_p$*, if we let* $\mathrm{Fr}_{\mathfrak{p}}$ *act on* $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})$ *as in Subsection 3.12, and on the spectral sequence* $(E_1^{-i,j},d_1^{-i,j})$ *as follows: for* $\mathtt{T}=\{\tau_1,\cdots,\tau_{\#\mathtt{T}}\}$*, we define*

$$\begin{aligned}\mathrm{Fr}_{\mathfrak{p}}:H^{j-2i}_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp,\mathscr{D}^{(\underline{k},w)}|_{Y_{\mathtt{T}}})e_{\mathtt{T}}&\longrightarrow H^{j-2i}_{c,\mathrm{rig}}(Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T}}/L_\wp,\mathscr{D}^{(\underline{k},w)}|_{Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T}}})e_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T}}\\ m_{\mathtt{T}}\cdot e_{\tau_1}\wedge\cdots\wedge e_{\tau_{\#\mathtt{T}}}&\longmapsto \mathrm{Fr}_{\mathfrak{p}}(m_{\mathtt{T}})\cdot p^{\#\mathtt{T}_{\mathfrak{p}}}\cdot e_{\sigma_{\mathfrak{p}}^{-1}\tau_1}\wedge\cdots\wedge e_{\sigma_{\mathfrak{p}}^{-1}\tau_{\#\mathtt{T}}},\end{aligned}$$

where $\mathtt{T}_{\mathfrak{p}}=\mathtt{T}\cap\Sigma_{\infty/\mathfrak{p}}$*. Similarly, the spectral sequence is equivariant for the action of* $S_{\mathfrak{p}}$ *and* $\Phi_{\mathfrak{p}^2}=\mathrm{Fr}_{\mathfrak{p}}^2\cdot S_{\mathfrak{p}}^{-1}$ *on both sides.*

Proof. — We put $\mathscr{F}=\mathscr{F}^{(\underline{k},w)}$ to simplify the notation. For any open subset $U\subseteq X^{\mathrm{tor}}$, let $j_U:U\hookrightarrow X^{\mathrm{tor}}$ denote the canonical immersion. By Lemma 4.3, we have the following right resolution of $j^\dagger_{X^{\mathrm{tor,ord}}}\mathrm{DR}^\bullet_c(\mathscr{F})$:
(4.10.2)

$$j^\dagger_{X^{\mathrm{tor}}-Y_{\Sigma_\infty}}\mathrm{DR}^\bullet_c(\mathscr{F})e_{\Sigma_\infty}\to\bigoplus_{\tau\in\Sigma_\infty}j^\dagger_{X^{\mathrm{tor}}-Y_{\Sigma_\infty\backslash\tau}}\mathrm{DR}^\bullet_c(\mathscr{F})e_{\Sigma_\infty\backslash\tau}\to\cdots\to\bigoplus_{\tau\in\Sigma_\infty}j^\dagger_{X^{\mathrm{tor}}-Y_\tau}\mathrm{DR}^\bullet_c(\mathscr{F})e_\tau.$$

Here, the $(-i)$-th term is a direct sum, over a subset $\mathtt{S}\subseteq\Sigma_\infty$ such that $\#\mathtt{T}=i$, of $j^\dagger_{X^{\mathrm{tor}}-Y_{\mathtt{T}}}\mathrm{DR}^\bullet_c(\mathscr{F})e_{\mathtt{T}}$; and all connection maps are natural restrictions. Hence, using the remark at the end of Subsection 4.2, we see that the following double complex, denoted by $K^{\bullet,\bullet}$,

$$\begin{array}{ccccccc}\mathrm{DR}^\bullet_c(\mathscr{F})e_{\Sigma_\infty}&\longrightarrow&\bigoplus_{\tau\in\Sigma_\infty}\mathrm{DR}^\bullet_c(\mathscr{F})e_{\Sigma_\infty\backslash\tau}&\longrightarrow\cdots\longrightarrow&\bigoplus_{\tau\in\Sigma_\infty}\mathrm{DR}^\bullet_c(\mathscr{F})e_\tau\\ \downarrow&&\downarrow&&\downarrow\\ j^\dagger_{X^{\mathrm{tor}}-Y_{\Sigma_\infty}}\mathrm{DR}^\bullet_c(\mathscr{F})e_{\Sigma_\infty}&\to&\bigoplus_{\tau\in\Sigma_\infty}j^\dagger_{X^{\mathrm{tor}}-Y_{\Sigma_\infty\backslash\tau}}\mathrm{DR}^\bullet_c(\mathscr{F})e_{\Sigma_\infty\backslash\tau}&\to\cdots\to&\bigoplus_{\tau\in\Sigma_\infty}j^\dagger_{X^{\mathrm{tor}}-Y_\tau}\mathrm{DR}^\bullet_c(\mathscr{F})e_\tau\end{array}$$

is quasi-isomorphic to

$$\mathrm{Cone}\big[\mathrm{DR}^\bullet_c(\mathscr{F})\to j^\dagger_{X^{\mathrm{tor,ord}}}\mathrm{DR}^\bullet_c(\mathscr{F})\big][-1].$$

In other words, if $\underline{s}(K^{\bullet,\bullet})$ denote the simple complex associated to $K^{\bullet,\bullet}$, then we have a quasi-isomophism

$$\Big(\underline{s}(K^{\bullet,\bullet})\to\mathrm{DR}^\bullet_c(\mathscr{F})\Big)\xrightarrow{\sim}j^\dagger_{X^{\mathrm{tor,ord}}}\mathrm{DR}^\bullet_c(\mathscr{F}), \tag{4.10.3}$$

where $\underline{s}(K^{\bullet,\bullet})\to\mathrm{DR}^\bullet_c(\mathscr{F})$ is induced by the sum of identity maps $\bigoplus_{\tau\in\Sigma_\infty}\mathrm{DR}^\bullet_c(\mathscr{F})e_\tau\to\mathrm{DR}^\bullet_c(\mathscr{F})$. Taking global sections on $\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}$, one obtains a spectral sequence in the second quadrant:

$$E_1^{-i,j}\Rightarrow H^{j-i}(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}},j^\dagger_{X^{\mathrm{tor,ord}}}\mathrm{DR}^\bullet_c(\mathscr{F}))=H^{j-i}_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}), \tag{4.10.4}$$

where $E_1^{0,j} = H^j(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}, \mathrm{DR}^\bullet_c(\mathscr{F}))$ and

$$E_1^{-i,j} = \bigoplus_{\#\mathtt{T}=i} H^j(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}, \mathrm{Cone}\left[\mathrm{DR}^\bullet_c(\mathscr{F}) \to j^\dagger_{X^{\mathrm{tor}}-Y_{\mathtt{T}}}\mathrm{DR}^\bullet_c(\mathscr{F})\right][-1])e_{\mathtt{T}}, \quad \text{for } i \geq 1.$$

We observe that, for each non-empty subset $\mathtt{T} \subseteq \Sigma_\infty$, we have a quasi-isomorphism of complexes

$$\mathrm{Cone}\left[\mathrm{DR}^\bullet_c(\mathscr{F}) \to j^\dagger_{X^{\mathrm{tor}}-Y_{\mathtt{T}}}\mathrm{DR}^\bullet_c(\mathscr{F})\right][-1] \simeq \mathrm{Cone}\left[j^\dagger_X\mathrm{DR}^\bullet(\mathscr{F}) \to j^\dagger_{X-Y_{\mathtt{T}}}\mathrm{DR}^\bullet(\mathscr{F})\right][-1]$$

by the excision. After taking global sections over $\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}$, one obtains the rigid cohomology with support in $Y_{\mathtt{T}}$:

$$R\Gamma_{Y_{\mathtt{T}},\mathrm{rig}}(X/L_\wp, \mathscr{D}^{(\underline{k},w)}) = R\Gamma(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}, \mathrm{Cone}\left[\mathrm{DR}^\bullet_c(\mathscr{F}) \to j^\dagger_{X^{\mathrm{tor}}-Y_{\mathtt{T}}}\mathrm{DR}^\bullet_c(\mathscr{F})\right][-1]).$$

Therefore, via the Gysin isomorphism (4.1.2), the term $E_1^{-i,j}$ in (4.10.4) for $i \geq 1$ is isomorphic to the direct sum of $H^{j-2i}_{Y_{\mathtt{T}},\mathrm{rig}}(X, \mathscr{D}^{(\underline{k},w)})$ for all $\mathtt{T} \subseteq \Sigma_\infty$ with $\#\mathtt{T} = i$.

Now statement (1) of the Proposition follows from (4.10.4) and Lemma 4.11 below. By functoriality of the construction, the spectral sequence is clearly equivariant under the action of $\mathscr{H}(K^p, L_\wp)$. For the equivariance under the actions of $\mathrm{Fr}_{\mathfrak{p}}$ and $\Phi_{\mathfrak{p}^2}$, it suffices to note that the action of $\mathrm{Fr}_{\mathfrak{p}}$ on the spectral sequence (4.10.1) has already taken account of the Frobenius twist given by the Gysin isomorphism. □

Lemma 4.11. — *The cohomology $H^\star(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}, \mathrm{DR}^\bullet_c(\mathscr{F}))$ is canonically isomorphic to the rigid cohomology with compact support $H^\star_{c,\mathrm{rig}}(X/L_\wp, \mathscr{D}^{(\underline{k},w)})$.*

Proof. — This is well-known to the experts, but unfortunately it is not well written down in the literature. To give a sketch of a short proof, we can pass to the dual. By [**BC94**, 2.6], we have an isomorphism $H^\star(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}, \mathrm{DR}^\bullet(\mathscr{F}^\vee)) \cong H^\star_{\mathrm{rig}}(X/L_\wp, \mathscr{D}^{(\underline{k},w),\vee})$. By [**Ke06**, Theorem 1.2.3], $H^\star_{c,\mathrm{rig}}(X/L_\wp, \mathscr{D}^{(\underline{k},w)})$ is in natural Poicaréd duality with $H^{2g-\star}_{\mathrm{rig}}(X/L_\wp, \mathscr{D}^{(\underline{k},w),\vee})$.

The proof of the Lemma will be finished, if we can show that $H^\star(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}, \mathrm{DR}^\bullet_c(\mathscr{F}))$ and $H^{2g-\star}(\mathfrak{X}^{\mathrm{tor}}_{\mathrm{rig}}, \mathrm{DR}^\bullet(\mathscr{F}^\vee))$ are in natural Poincaré duality. By rigid GAGA theorems, it is the same thing to prove that the algebraic de Rham cohomology $H^\star(\mathbf{X}^{\mathrm{tor}}_{L_\wp}, \mathrm{DR}^\bullet_c(\mathscr{F}))$ and $H^{2g-\star}(\mathbf{X}^{\mathrm{tor}}_{L_\wp}, \mathrm{DR}^\bullet(\mathscr{F}^\vee))$ are in Poincaré duality. Unfortunately, this is only available in the literature [**BCF04**] when $\mathscr{F}$ equals to the constant sheaf. One can either modifies the proof of *loc. cit.* for the general case; or alternatively, using the embedding $L_\wp \hookrightarrow \overline{\mathbb{Q}}_p \xleftarrow{\sim} \mathbb{C}$, one reduces to show that $H^\star(\mathbf{X}^{\mathrm{tor}}(\mathbb{C}), \mathrm{DR}^\bullet_c(\mathscr{F}))$ and $H^{2g-\star}(\mathbf{X}^{\mathrm{tor}}(\mathbb{C}), \mathrm{DR}^\bullet(\mathscr{F}^\vee))$ are in Poincaré duality. Let $\mathbb{L} = (\mathscr{F}|_{\mathbf{X}(\mathbb{C})})^{\nabla=0}$ denote the local system of horizontal sections of $\mathscr{F}$ on $\mathbf{X}(\mathbb{C})$. By the Riemann-Hilbert-Deligne correspondence and classical GAGA, $H^{2g-\star}(\mathbf{X}^{\mathrm{tor}}(\mathbb{C}), \mathrm{DR}^\bullet(\mathscr{F}^\vee))$ is canonically isomorphic to the singular cohomology $H^{2g-\star}(\mathbf{X}(\mathbb{C}), \mathbb{L}^\vee)$. By the same arguments as in [**FC90**, Chap. VI 5.4], one sees that $H^\star(\mathbf{X}^{\mathrm{tor}}(\mathbb{C}), \mathrm{DR}^\bullet_c(\mathscr{F}))$ is the same as $H^\star(\mathbf{X}^{\mathrm{tor}}(\mathbb{C}), j_!(\mathbb{L})) = H^\star_c(\mathbf{X}(\mathbb{C}), \mathbb{L})$, where $j : \mathbf{X}(\mathbb{C}) \to \mathbf{X}^{\mathrm{tor}}(\mathbb{C})$ denotes the natural immersion. The desired duality now follows from the classical Poincaré theory for manifolds. □

4.12. Étale Cohomology. — In order to compute the cohomology groups $H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)})$, we compare it with its étale analogues. Let $l \neq p$ be a fixed prime, and fix an isomorphism $\iota_l : \mathbb{C} \xrightarrow{\simeq} \overline{\mathbb{Q}}_l$. This defines an l-adic place of the number field L; denote by $L_{\mathfrak{l}}$ its completion. Post-composition with ι_l identifies Σ_∞ with the l-adic embeddings of F. Let $a : \mathcal{A} \to \mathbf{Sh}_K(G)$ be the structural morphism of the universal abelian scheme. The relative étale cohomology $R^1a_*(L_{\mathfrak{l}})$ has a canonical decomposition:

$$R^1a_*(L_{\mathfrak{l}}) = \bigoplus_{\tau\in\Sigma_\infty} R^1a_*(L_{\mathfrak{l}})_\tau,$$

where $R^1a_*(L_{\mathfrak{l}})_\tau$ is the direct summand on which F acts via $\iota_l\circ\tau$. For a multiweight $(\underline{k},w)$, we put

$$\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}:=\bigotimes_{\tau\in\Sigma_\infty}\Big((\wedge^2R^1a_*(L_{\mathfrak{l}})_\tau)^{\frac{w-k_\tau}{2}}\otimes\mathrm{Sym}^{k_\tau-2}R^1a_*(L_{\mathfrak{l}})_\tau\Big).$$

Note that $\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}|_X$ is a lisse $L_{\mathfrak{l}}$-sheaf pure of weight $g(w-2)$. We have a natural action of the prime-to-p Hecke operators $\mathscr{H}(K^p,L_{\mathfrak{l}})$ on $H^\star_{\mathrm{et}}(Y_{\mathtt{T}},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})$ for each $\mathtt{T}\subseteq X$. For each $\mathfrak{p}\in\Sigma_p$, consider $\mathrm{Fr}_{\mathfrak{p}}:X\to X$. The isogeny $\pi_{\mathfrak{p}}:\mathcal{A}\to\varphi_{\mathfrak{p}}^*\mathcal{A}$ induces an isomorphism $\pi_{\mathfrak{p}}^*\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}\xrightarrow{\simeq}\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}$. This gives rise to an action of the partial Frobenius on cohomology groups:

$$(4.12.1)\quad \mathrm{Fr}_{\mathfrak{p}}:H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})\xrightarrow{\varphi_{\mathfrak{p}}^*}H^\star_{c,\mathrm{et}}(Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T},\overline{\mathbb{F}}_p},\varphi_{\mathfrak{p}}^*\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})\xrightarrow{\pi_{\mathfrak{p}}^*}H^\star_{c,\mathrm{et}}(Y_{\sigma_{\mathfrak{p}}^{-1}\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}).$$

As usual, we put $Y_\emptyset=X$. Similarly, we have morphisms $\Phi_{\mathfrak{p}^2}$ and $S_{\mathfrak{p}}$ on $H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})$, as in the case of rigid cohomology. For simplicity, we use $Y^{(i)}$ to denote the *disjoint union* of the GO-strata $Y_{\mathtt{T}}$ for $\#\mathtt{T}=i$. Then, for a fixed integer $i\geq 0$, $H^\star_{c,\mathrm{et}}(Y^{(i)}_{\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})=\bigoplus_{\#\mathtt{T}=i}H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})$ and its rigid analogue is stable under the action of $\mathrm{Fr}_{\mathfrak{p}}$ for each $\mathfrak{p}$.

Proposition 4.13. — *We identify both $\overline{\mathbb{Q}}_l$ and $\overline{\mathbb{Q}}_p$ with $\mathbb{C}$ using the isomorphisms ι_l and ι_p. Then for an integer $i\geq 0$, we have an equality in the Grotendieck group of modules over $\mathscr{H}(K^p,\mathbb{C})[\mathrm{Fr}_{\mathfrak{p}},S_{\mathfrak{p}},S_{\mathfrak{p}}^{-1};\mathfrak{p}\in\Sigma_p]$:*

$$\sum_{n=0}^{2g-2i}(-1)^n[\bigoplus_{\#\mathtt{T}=i}H^n_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})\otimes_{L_{\mathfrak{l}}}\overline{\mathbb{Q}}_l]=\sum_{n=0}^{2g-2i}(-1)^n[\bigoplus_{\#\mathtt{T}=i}H^n_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_{\wp},\mathscr{D}^{(\underline{k},w)})\otimes_{L_{\wp}}\overline{\mathbb{Q}}_p].$$

Moreover, if $i\neq 0$, we have an equality for each n:

$$[\bigoplus_{\#\mathtt{T}=i}H^n_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})\otimes_{L_{\mathfrak{l}}}\overline{\mathbb{Q}}_l]=[\bigoplus_{\#\mathtt{T}=i}H^n_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_{\wp},\mathscr{D}^{(\underline{k},w)})\otimes_{L_{\wp}}\overline{\mathbb{Q}}_p]$$

.

Proof. — Note that $\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}$ and $\mathscr{D}^{(\underline{k},w)}$ on X are pure of weight $g(w-2)$ in the sense of Deligne and [**AC13**] respectively. As $Y_{\mathtt{T}}$ is proper and smooth for $\mathtt{T}\neq\emptyset$, $H^n_{\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})$ and $H^n_{\mathrm{rig}}(Y_{\mathtt{T}}/L_{\wp},\mathscr{D}^{(\underline{k},w)})$ are both pure of weight $g(w-2)+n$ by Deligne's Weil II and its rigid analogue (*loc. cit.*). Since $\prod_{\mathfrak{p}\in\Sigma_p}\mathrm{Fr}_{\mathfrak{p}}$ is the Frobenius endomorphism of X, the weight can be detected by the action of $\prod_{\mathfrak{p}\in\Sigma_p}\mathrm{Fr}_{\mathfrak{p}}$, the second part of the Proposition follows immediately from the first part.

To prove the first part, we follow the strategy of [**Sa09**, §6]. We consider $F\otimes_{\mathbb{Q}}L\simeq\prod_{\tau\in\Sigma_\infty}L_\tau$, where L_τ is the copy of L with embedding $\tau:F\hookrightarrow L$. Let $e_\tau\in F\otimes_{\mathbb{Q}}L$ denote the projection onto L_τ. Since $F\otimes_{\mathbb{Q}}L$ is generated over L by $1+p\mathcal{O}_F$, we may write e_τ as a linear combination of elements in $1+p\mathcal{O}_F$. Hence, e_τ is a linear combination of endomorphisms of $\mathcal{A}$ over $\mathbf{X}$ of degrees prime to p.

Using the Vandermonde determinant, one can find easily a $\mathbb{Q}$-linear combination e^1 of multiplications by prime-to-p integers on $\mathcal{A}$ such that the induced action of e^1 on $R^1a_*(\mathbb{Q}_l)$ is identity, and is 0 on $R^qa_*\mathbb{Q}_l$ for $q\neq 1$. Consider the fiber product $a^{w-2}:\mathcal{A}^{w-2}\to\mathbf{X}$. Consider the fiber product $a^{w-2}:\mathcal{A}^{w-2}\to\mathbf{X}$. Then $e_\tau^{\otimes w-2}\cdot(e^1)^{\otimes w-2}$ acts as an idemponent on $R^qa_*^{w-2}(L_{\mathfrak{l}})$, we get $(R^1a_*(L_{\mathfrak{l}})_\tau)^{\otimes(w-2)}$ if $q=w-2$, and 0 if $q\neq w-2$. One finds also easily an idemponent $e^{(k_\tau,w)}\in\mathbb{Q}[\mathfrak{S}_{w-2}]$ in the group algebra of the symmetric group with $w-2$ letters such that

$$e^{(k_\tau,w)}\cdot(R^1a_*(L_{\mathfrak{l}})_\tau)^{\otimes(w-2)}=(\wedge^2R^1a_*(L_{\mathfrak{l}})_\tau)^{\frac{w-k_\tau}{2}}\otimes\mathrm{Sym}^{k_\tau-2}R^1a_*(L_{\mathfrak{l}})_\tau.$$

Note the action of $\mathfrak{S}_{w-2}$ on $R^1a_*(L_{\mathfrak{l}})_\tau^{\otimes(w-2)}$ is induced by its action on $\mathcal{A}^{w-2}$.

Consider the fiber product $a^{(w-2)g} : \mathcal{A}^{(w-2)g} \to \mathbf{X}$. Taking the product of the previous constructions, we get a L-linear combination $e^{(\underline{k},w)}$ of algebraic correspondences on $\mathcal{A}^{(w-2)g}$ satisfying the following properties:

1. It is an L-linear combination of permutations in $\mathfrak{S}_{(w-2)g}$ and endomorphisms of $\mathcal{A}^{(w-2)g}$ an an abelian scheme over $\mathbf{X}$ whose degrees are prime to p,
2. The action of $e^{(\underline{k},w)}$ on the cohomology $R^q a_*^{(w-2)g}(L_{\mathfrak{l}})$ is the projection onto the direct summand $\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}$ if $q = (w-2)g$ and is equal to 0 if $q \neq (w-2)g$.

The algebraic correspondence $e^{(\underline{k},w)}$ with coefficients in L acts also on $H^q_{c,\mathrm{et}}(\mathcal{A}^{g(w-2)}_{\overline{\mathbb{F}}_p}, L_{\mathfrak{l}})$. Using Leray spectral sequence for $a^{(w-2)g}$, one sees easily that, for any locally closed subscheme $Z \subseteq X$, we have

$$e^{(\underline{k},w)} \cdot H^{n+(w-2)g}_{c,\mathrm{et}}(\mathcal{A}^{g(w-2)} \times_X Z_{\overline{\mathbb{F}}_p}, L_{\mathfrak{l}}) = H^n_{c,\mathrm{et}}(Z_{\overline{\mathbb{F}}_p}, \mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}). \tag{4.13.1}$$

Similarly, let $(\mathcal{A}^{(w-2)g}/W(k_0))_{\mathrm{cris}}$ over $(X/W(k_0))_{\mathrm{cris}}$ denote respectively the small crystalline sites of $\mathcal{A}^{(w-2)g}$ and X with respect to $W(k_0)$, and $R^q_{\mathrm{cris}} a_*^{(w-2)g}(\mathcal{O}_{\mathcal{A}^{(w-2)g}/W(k_0)})$ be the relative crystalline cohomology. This is an F-crystal over $(X/W(k_0))_{\mathrm{cris}}$, and we denote by $\mathcal{H}^q_{\mathrm{rig}}(\mathcal{A}^{(w-2)g}/X)$ the associated overconvergent isocrystal on $X/L_{\wp}$. The algebraic correspondence $e^{(\underline{k},w)}$ acts on $\mathcal{H}^q_{\mathrm{rig}}(\mathcal{A}^{(w-2)g}/X)$ as an idempotent, and we have

$$e^{(\underline{k},w)} \cdot \mathcal{H}^q_{\mathrm{rig}}(\mathcal{A}^{(w-2)g}/X) = \begin{cases} \mathscr{D}^{(\underline{k},w)} & \text{if } q = (w-2)g, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$e^{(\underline{k},w)} \cdot H^{n+(w-2)g}_{c,\mathrm{rig}}(\mathcal{A}^{g(w-2)} \times_X Z/L_{\wp}) = H^n_{c,\mathrm{rig}}(Z/L_{\wp}, \mathscr{D}^{(\underline{k},w)})$$

for any subscheme $Z \subseteq X$.

As K^p varies, the Hecke action of $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$ on $\mathbf{Sh}_K(G)$ (2.10.2) lifts to an equivariant action on $\mathcal{A}$. Then, for each double coset $[K^p g K^p]$ with $g \in \mathrm{GL}_2(\mathbb{A}^{\infty,p})$, there exists a finite flat algebraic correspondence on $\mathcal{A}^{(w-2)g}$ such that, after composition with $e^{(\underline{k},w)}$, its induced actions on $H^{(w-2)g+n}_{c,\mathrm{et}}(\mathcal{A}^{(w-2)g} \times_X Y_{\mathtt{T},\overline{\mathbb{F}}_p}, L_{\mathfrak{l}})$ and $H^{n+(w-2)g}_{c,\mathrm{rig}}(\mathcal{A}^{g(w-2)} \times_X Y_{\mathtt{T}}/L_{\wp})$ give the Hecke action of $[K^p g K^p]$ on $H^n_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p}, \mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})$ and $H^n_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_{\wp}, \mathscr{D}^{(\underline{k},w)})$, respectively.

Consider the partial Frobenius $\varphi_{\mathfrak{p}} : X \to X$ for each $\mathfrak{p} \in \Sigma_p$. The isogeny $\pi_{\mathfrak{p}}^{(w-2)g}$: $\mathcal{A}^{(w-2)g} \to \varphi_{\mathfrak{p}}^*(\mathcal{A}^{(w-2)g}) = \varphi_{\mathfrak{p}}^*(\mathcal{A})^{(w-2)g}$ defines an algebraic correspondence on $\mathcal{A}^{(w-2)g}$ whose composition with $e^{(\underline{k},w)}$ induces the action of $\mathrm{Fr}_{\mathfrak{p}}$ on $H^n_{c,\mathrm{et}}(Y^{(i)}_{\overline{\mathbb{F}}_p}, \mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})$ and on $H^n_{c,\mathrm{rig}}(Y^{(i)}/L_{\wp}, \mathscr{D}^{(\underline{k},w)})$. Similarly, we see also that the action of $S_{\mathfrak{p}}$ and $S_{\mathfrak{p}}^{-1}$ on the étale and rigid cohomology groups of $Y^{(i)}$ are also induced by algebraic correspondences on $\mathcal{A}^{(w-2)g}$.

In summary, the action of $\mathscr{H}(K^p, L)[\mathrm{Fr}_{\mathfrak{p}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1}; \mathfrak{p} \in \Sigma_p]$ on the étale and rigid cohomolgy groups are linear combinations of actions induced by algebraic correspondences on $\mathcal{A}^{(w-2)g}$. Therefore, in order to prove the first part of the Proposition, it suffices to show, for any algebraic correspondence Γ of $\mathcal{A}^{(w-2)g}$ and an integer $i \geq 0$, we have the following equality:

$$\sum_n (-1)^n \mathrm{Tr}(\Gamma^*, H^n_{c,\mathrm{et}}(\mathcal{A}^{(w-2)g} \times_X Y^{(i)}_{\overline{\mathbb{F}}_p}, L_{\mathfrak{l}})) = \sum_n (-1)^n \mathrm{Tr}(\Gamma^*, H^n_{c,\mathrm{rig}}(\mathcal{A}^{(w-2)g} \times_X Y^{(i)}/L_{\wp})).$$

If $i \geq 1$, $\mathcal{A}^{(w-2)g} \times_X Y^{(i)}$ is proper and smooth over k_0. Since the cycle class map is well defined for étale and rigid cohomology [**Pe03**], the Lefschetz formula is valid and the both sides are equal to the intersection number (Γ, Δ), where Δ is the diagonal of $(\mathcal{A}^{(w-2)g} \times_X Y^{(i)}) \times (\mathcal{A}^{(w-2)g} \times_X Y^{(i)})$. If $i = 0$, the desired equality still holds thanks to [**Mi09**, Corollary 3.3], whose proof uses Fujiwara's trace formula [**Fu97**] and its rigid analogue due to Mieda. □

Remark 4.14. — Even though it will not be used in this paper, it is interesting to consider the étale counterpart of $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$. Let $t: X \hookrightarrow X^{\mathrm{tor}}$ and $j: X^{\mathrm{tor,ord}} \to X^{\mathrm{tor}}$ be the natural open immersions. Then $X^{\mathrm{tor,ord}}$ can be viewed as a partial compactification of the ordinary locus X^{ord}. We consider the cohomology group $H^\star_{\mathrm{et}}(X^{\mathrm{tor,ord}}_{\overline{\mathbb{F}}_p}, t_!(\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}|_{X^{\mathrm{ord}}})) = H^\star_{\mathrm{et}}(X^{\mathrm{tor}}_{\overline{\mathbb{F}}_p}, Rj_*t_!(\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}|_{X^{\mathrm{ord}}}))$. Similarly to the rigid case, it is equipped with a natural action of the algebra $\mathscr{H}(K^p, L_{\mathfrak{l}})[\mathrm{Fr}_{\mathfrak{p}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1}; \mathfrak{p} \in \Sigma_p]$. Using the cohomological purity for smooth pairs [SGA 4, XVI Théorème 3.3], it is easy to prove that

$$R^b j_* t_!(\mathscr{L}^{(\underline{k},w)}|_{X^{\mathrm{ord}}}) = \begin{cases} t_!(\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}) & \text{if } b = 0, \\ \bigoplus_{\#\mathtt{T}=b} \mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}|_{Y_{\mathtt{T}}}(-b), & \text{if } b \geq 1. \end{cases}$$

One deduces immediately a spectral sequence

$$E_2^{a,b} = H^a_{\mathrm{et}}(X^{\mathrm{tor}}_{\overline{\mathbb{F}}_p}, R^b j_* t_!(\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})) = \bigoplus_{\#\mathtt{T}=b} H^a_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p}, \mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)}|_{Y_{\mathtt{T}}})(-b) \Rightarrow H^{a+b}_{\mathrm{et}}(X^{\mathrm{tor,ord}}_{\overline{\mathbb{F}}_p}, t_!(\mathscr{L}_{\mathfrak{l}}^{\underline{k},w})),$$

which is $\mathscr{H}(K^p, L_{\mathfrak{l}})[\mathrm{Fr}_{\mathfrak{p}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1}; \mathfrak{p} \in \Sigma_p]$-equivariant if we define an action of $\mathscr{H}(K^p, L_{\mathfrak{l}})$, $\mathrm{Fr}_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$ on the left hand side in a similar way as its rigid analogue (4.10.1). Then by Proposition 4.13, we have an equality in the Grothendieck group of modules over $\mathscr{H}(K^p, \mathbb{C})[\mathrm{Fr}_{\mathfrak{p}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1}; \mathfrak{p} \in \Sigma_p]$:

$$\sum_n (-1)^n [H^n_{\mathrm{et}}(X^{\mathrm{tor,ord}}_{\overline{\mathbb{F}}_p}, t_!(\mathscr{L}_{\mathfrak{l}}^{(\underline{k},w)})) \otimes_{L_{\mathfrak{l}}} \overline{\mathbb{Q}}_l] = \sum_n (-1)^n [H^n_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)}) \otimes_{L_{\wp}} \overline{\mathbb{Q}}_p].$$

As usual, we identify both $\overline{\mathbb{Q}}_l$ and $\overline{\mathbb{Q}}_p$ with $\mathbb{C}$ via ι_l and ι_p, respectively.

5. Quaternionic Shimura Varieties and Goren-Oort Stratification

As shown in Proposition 4.10, the cohomology group $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ can be computed by a spectral sequence consisting of rigid cohomology on the GO-strata of the (special fiber of) Hilbert modular variety; computing this is further equivalent to computing the étale counterparts, as shown in Proposition 4.13. The aim of this section is to compute the corresponding étale cohomology groups together with the actions of various operators. The first step is to relate the étale cohomology of the GO-strata to the étale cohomology of analogous GO-strata of the Shimura variety for the group $G''_\emptyset = \mathrm{GL}_{2,F} \times_{F^\times} E^\times$ for certain CM extension E of F (Proposition 5.16). The next step is to apply the main theorem in the previous paper [**TX13a**] of this sequel which states that each such a GO-stratum is isomorphic to a $\mathbb{P}^1$-power bundle over some other Shimura varieties (Theorem 5.22). Combining these two, we arrive at an explicit description of those étale cohomology groups appearing in Proposition 4.13 (which contains similar information as each term of the spectral sequence (4.10.1) does); this is done in Propositions 5.20 and 5.24. One subtlety is that, in general, we do not have full information on the action of twisted partial Frobenius on these spaces (Conjecture 5.18). This is why a complete description is only available when p is inert. Nonetheless, we can still prove our main theorem on classicality, as shown in the next section.

This section will start with a long digression on introducing quaternionic Shimura varities and certain unitary-like Shimura varieties; the reason for this detour is that the description of the GO-strata does naturally live over the special fiber of Hilbert modular varieties but rather the unitary-like ones.

5.1. Quaternionic Shimura variety. — We shall only recall the quaternionic Shimura varieties that we will need. For more details, see [**TX13a**, §3]. Let $\mathtt{S}$ be an even subset of places of F. Put $\mathtt{S}_\infty = \mathtt{S} \cap \Sigma_\infty$. We denote by $B_{\mathtt{S}}$ be the quaternionic algebra over F ramified exactly at $\mathtt{S}$, and $G_{\mathtt{S}} = \mathrm{Res}_{F/\mathbb{Q}}(B_{\mathtt{S}}^\times)$ be the associated $\mathbb{Q}$-algebraic group. We consider the homomophism

$$h_{\mathtt{S}} : \mathbb{C}^\times \to G_{\mathtt{S}}(\mathbb{R}) \simeq (\mathbb{H}^\times)^{\mathtt{S}_\infty} \times \mathrm{GL}_2(\mathbb{R})^{\Sigma_\infty - \mathtt{S}_\infty}$$

given by $h_{\mathtt{S}}(x+yi) = (z_{G_{\mathtt{S}}}^\tau)_{\tau\in\Sigma_\infty}$ such that $z_{G_{\mathtt{S}}}^\tau = 1$ for $\tau \in \mathtt{S}_\infty$ and $z_{G_{\mathtt{S}}}^\tau = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ for $\tau \in \Sigma_\infty - \mathtt{S}_\infty$. Then the $G_{\mathtt{S}}(\mathbb{R})$-conjugacy class of $h_{\mathtt{S}}$ is isomorphic to $\mathfrak{H}_{\mathtt{S}} = (\mathfrak{h}^{\pm})^{\Sigma_\infty - \mathtt{S}_\infty}$, where $\mathfrak{h}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. For an open compact subgroup $K_{\mathtt{S}} \subset G_{\mathtt{S}}(\mathbb{A}^\infty)$, we put

$$\mathrm{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})(\mathbb{C}) = G_{\mathtt{S}}(\mathbb{Q})\backslash \mathfrak{H}_{\mathtt{S}} \times G_{\mathtt{S}}(\mathbb{A}^\infty)/K_{\mathtt{S}}.$$

The Shimura variety $\mathrm{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$ is defined over its reflex field $F_{\mathtt{S}}$, which is a subfield of the Galois closure of F in $\mathbb{C}$. We have a natural action of the group $G_{\mathtt{S}}(\mathbb{A}^\infty)$ on $\mathrm{Sh}(G_{\mathtt{S}}) = \varprojlim_{K_{\mathtt{S}}} \mathrm{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$. When $\mathtt{S} = \emptyset$, this is the Hilbert modular varieties $\mathrm{Sh}(G)$ considered in Section 2.

For each $\mathfrak{p}$, we put $\mathtt{S}_{\infty/\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}} \cap \mathtt{S}$. In this paper, we will consider only $\mathtt{S}$ satisfying

Hypothesis 5.2. — We have $\mathtt{S} \subseteq \Sigma_\infty \cup \Sigma_p$. (Put $\mathtt{S}_p = \mathtt{S}\cap\Sigma_p$.) Moreover, for a p-adic place $\mathfrak{p} \in \Sigma_p$,

1. if $\mathfrak{p} \in \mathtt{S}$, then the degree $[F_\mathfrak{p} : \mathbb{Q}_p]$ is odd and $\Sigma_{\infty/\mathfrak{p}} \subseteq \mathtt{S}$;
2. if $\mathfrak{p} \notin \mathtt{S}$, then $\mathtt{S}_{\infty/\mathfrak{p}}$ has even cardinality.

We fix an isomorphism $G_{\mathtt{S}}(\mathbb{A}^{\infty,p}) \simeq \mathrm{GL}_2(\mathbb{A}_F^{\infty,p})$, so that the prime-to-$p$ component $K_{\mathtt{S}}^p$ will be considered as an open subgroup of $\mathrm{GL}_2(\mathbb{A}_F^{\infty,p})$. In this paper, we will only encounter primes $\mathfrak{p} \in \Sigma_p$ and open compact subgroups $K_{\mathtt{S},\mathfrak{p}} \subset B_{\mathtt{S}}^\times(F_\mathfrak{p})$ of the following types:

- **Type** α **and** $\alpha^\#$ The degree $[F_\mathfrak{p} : \mathbb{Q}_p]$ is even, so $B_{\mathtt{S}}$ splits at $\mathfrak{p}$ by Hypothesis 5.2. Fix an isomorphism $B_{\mathtt{S}}^\times(F_\mathfrak{p}) \simeq \mathrm{GL}_2(F_\mathfrak{p})$. We will only consider $K_{\mathtt{S},\mathfrak{p}}$ to be either $\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ or the Iwahoric subgroup $I_\mathfrak{p} \subset \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ (3.3.1), and the latter case may only happen when $\Sigma_{\infty/\mathfrak{p}} = \mathtt{S}_{\infty/\mathfrak{p}}$. We will say $\mathfrak{p}$ is of *type* α if $K_{\mathtt{S},\mathfrak{p}} = \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ is hyperspecial, and is of *type* $\alpha^\#$ *if* $K_{\mathtt{S},\mathfrak{p}}$ is Iwahoric. Note that when $\Sigma_{\infty/\mathfrak{p}} \neq \mathtt{S}_{\infty/\mathfrak{p}}$, $\mathfrak{p}$ is necessarily of type α, but when $\Sigma_{\infty/\mathfrak{p}} = \mathtt{S}_{\infty/\mathfrak{p}}$, both types are possible.
- **Type** β **and** $\beta^\#$ The degree $[F_\mathfrak{p} : \mathbb{Q}_p]$ is odd. There are two cases:
  - When $\mathtt{S}_{\infty/\mathfrak{p}} \neq \Sigma_{\infty/\mathfrak{p}}$, $B_{\mathtt{S}}$ splits at $\mathfrak{p}$ by Hypothesis 5.2. We fix an isomorphism $B_{\mathtt{S}}^\times(F_\mathfrak{p}) \simeq \mathrm{GL}_2(F_\mathfrak{p})$. We consider only the case $K_{\mathtt{S},\mathfrak{p}} = \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$. We say $\mathfrak{p}$ is of *type* β.
  - When $\mathtt{S}_{\infty/\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}}$, $B_{\mathtt{S}}$ is ramified at $\mathfrak{p}$. Then $B_{\mathtt{S},\mathfrak{p}} := B_{\mathtt{S}} \otimes_F F_\mathfrak{p}$ is the quaternion division algebra over $F_\mathfrak{p}$. Let $\mathcal{O}_{B_{\mathtt{S},\mathfrak{p}}}$ be the maximal order of $B_{\mathtt{S},\mathfrak{p}}$. We will only allow $K_{\mathtt{S},\mathfrak{p}} = \mathcal{O}_{B_{\mathtt{S},\mathfrak{p}}}^\times$. We say $\mathfrak{p}$ is of *type* $\beta^\#$.

Let $K_{\mathtt{S}} = K_{\mathtt{S}}^p \cdot \prod_\mathfrak{p} K_{\mathtt{S},\mathfrak{p}} \subset G_{\mathtt{S}}(\mathbb{A}^\infty)$ be an open compact subgroup of the types considered above. The isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$ determines a p-adic place $\wp$ of the reflex field $F_{\mathtt{S}}$. Let $\mathcal{O}_\wp$ be the valuation ring of $F_{\mathtt{S},\wp}$, and $\wp$ its residue field.

Theorem 5.3 ([**TX13a**, Cor 3.18]). — *For $K_{\mathtt{S}}^p$ sufficiently small, there exists a smooth quasi-projective scheme $\mathbf{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$ over $\mathcal{O}_\wp$ such that*

$$\mathbf{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}}) \times_{\mathcal{O}_\wp} F_{\mathtt{S},\wp} \simeq \mathrm{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}}) \times_{F_{\mathtt{S}}} F_{\mathtt{S},\wp}.$$

If $\mathtt{S} = \emptyset$, then $\mathbf{Sh}_K(G)$ (we omit $\emptyset$ from the notation) is isomorphic to the integral models of Hilbert modular varieties considered in Subsection 2.3; if $\mathtt{S} \neq \emptyset$, $\mathbf{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$ is projective.

The construction of $\mathbf{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$ in *loc. cit.* makes uses of an auxiliary choice of CM extension E/F such that $B_{\mathtt{S}} \otimes_F E$ is isomorphic to $\mathrm{M}_2(E)$. (We fix such an isomorphism from now on.) When $p \geq 3$, the integral model $\mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}})$ satisfies certain extension property (see [**TX13a**, 2.4]) and hence does not depend on the choice of such E. The basic idea of the construction follows the method of “modèles étranges” of Deligne [**De71**] and Carayol

[**Ca86**], but we allow certain p-adic places of F to be inert in E so that the construction may be used to describe Goren-Oort strata.

5.4. Auxiliary CM extension. — Let E/F be a CM extension that is split over all $\mathfrak{p}\in\Sigma_p$ of type α or $\alpha^\#$ and is inert over all $\mathfrak{p}\in\Sigma_p$ of type β or $\beta^\#$. Denote by $\Sigma_{E,\infty}$ the set of archimedean embeddings of E, and $\Sigma_{E,\infty}\to\Sigma_\infty$ the natural two-to-one map given by restriction to F. Our construction depends on a choice of subset $\tilde{\mathtt{S}}_\infty$ consisting of, for each $\tau\in\mathtt{S}_\infty$, a choice of exactly one $\tilde\tau$ extending the archimedean embedding of F to E. This is equivalent to the collection of numbers $s_{\tilde\tau}\in\{0,1,2\}$ for each $\tilde\tau\in\Sigma_{E,\infty}$ such that

- if $\tau\in\Sigma_\infty-\mathtt{S}_\infty$, we have $s_{\tilde\tau}=1$ for all lifts $\tilde\tau$ of τ;
- if $\tau\in\mathtt{S}_\infty$ and $\tilde\tau$ is the lift in $\tilde{\mathtt{S}}_\infty$, we have $s_{\tilde\tau}=2$ and $s_{\tilde\tau^c}=0$, where $\tilde\tau^c$ is the conjugate of $\tilde\tau$.

Put $T_{E,\tilde{\mathtt{S}}}=\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$, where the subscript $\tilde{\mathtt{S}}=(\mathtt{S},\tilde{\mathtt{S}}_\infty)$ indicates that our choice of the homomorphism:

$$\begin{array}{rcl} h_{E,\tilde{\mathtt{S}}}\colon \mathbb{S}(\mathbb{R})=\mathbb{C}^\times & \longrightarrow & T_{E,\tilde{\mathtt{S}}}(\mathbb{R})=\bigoplus_{\tau\in\Sigma_\infty}(E\otimes_{F,\tau}\mathbb{R})^\times\simeq\bigoplus_{\tau\in\Sigma_\infty}\mathbb{C}^\times \\ z & \longmapsto & (z_{E,\tau})_\tau. \end{array}$$

Here $z_{E,\tau}=1$ if $\tau\notin\mathtt{S}_\infty$ and $z_{E,\tau}=z$ if $\tau\in\mathtt{S}_\infty$, in which case, the isomorphism $(E\otimes_{F,\tau}\mathbb{R})^\times\simeq\mathbb{C}^\times$ is given by the lift $\tilde\tau\in\tilde{\mathtt{S}}_\infty$. The reflex field $E_{\tilde{\mathtt{S}}}$ is the field of definition of the Hodge cocharacter

$$\mu_{E,\tilde{\mathtt{S}}}\colon \mathbb{G}_{m,\mathbb{C}}\xrightarrow{z\mapsto(z,1)}\mathbb{S}(\mathbb{C})\simeq\mathbb{C}^\times\times\mathbb{C}^\times\xrightarrow{h_{E,\tilde{\mathtt{S}}}}T_{E,\tilde{\mathtt{S}}}(\mathbb{C}),$$

where the first copy $\mathbb{C}^\times$ in $\mathbb{S}(\mathbb{C})$ is given by the identity character of $\mathbb{C}^\times$, and the second by complex conjugation. More precisely, $E_{\tilde{\mathtt{S}}}$ is the subfield of $\mathbb{C}$ corresponding to the subgroup of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ which stabilizes the set $\tilde{\mathtt{S}}_\infty$; it contains $F_{\mathtt{S}}$ as a subfield. The isomorphism $\iota_p:\mathbb{C}\simeq\overline{\mathbb{Q}}_p$ determines a p-adic place $\tilde\wp$ of $E_{\tilde{\mathtt{S}}}$; we use $\mathcal{O}_{\tilde\wp}$ to denote the valuation ring and $k_{\tilde\wp}$ the residue field.

We take the level structure K_E to be $K_E^pK_{E,p}$, where $K_{E,p}=(\mathcal{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times$, and K_E^p is an open compact subgroup of $\mathbb{A}_E^{\infty,p,\times}$. This then gives rise to a Shimura variety $\mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}}})$ and its limit $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})=\varprojlim_{K_E^p}\mathrm{Sh}_{K_{E,p}K_E^p}(T_{E,\tilde{\mathtt{S}}})$; they have integral models $\mathbf{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}}})$ and $\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})$ over $\mathcal{O}_{\tilde\wp}$. The set of $\mathbb{C}$-points of the limit is given by

$$\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})(\mathbb{C})=T_{E,\tilde{\mathtt{S}}}(\mathbb{Q})^{\mathrm{cl}}\backslash T_{E,\tilde{\mathtt{S}}}(\mathbb{A}^\infty)/K_{E,p}=\mathbb{A}_E^{\infty,p,\times}/\mathcal{O}_{E,(p)}^{\times,\mathrm{cl}},$$

where the superscript cl denotes the closure in the appropriate topological groups. The geometric Frobenius $\mathrm{Frob}_{\tilde\wp}$ in the Galois group $\mathrm{Gal}_{k_{\tilde\wp}}=\mathrm{Gal}(\mathcal{O}_{\tilde\wp}^{\mathrm{ur}}/\mathcal{O}_{\tilde\wp})$ acts on $\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})$ by multiplication by the image of local uniformizer at $\tilde\wp$ in the idèles of $E_{\tilde{\mathtt{S}}}$ under the following reciprocity map

$$\begin{aligned}\mathfrak{Rec}_E\colon \mathbb{A}_{E_{\tilde{\mathtt{S}}}}^{\infty,\times}/E_{\tilde{\mathtt{S}}}^\times\times\mathcal{O}_{\tilde\wp}^\times &\xrightarrow{\mu_{E,\mathtt{S}}} T_{E,\tilde{\mathtt{S}}}(\mathbb{A}_{E_{\tilde{\mathtt{S}}}}^\infty)/T_{E,\tilde{\mathtt{S}}}(E_{\tilde{\mathtt{S}}})T_{E,\tilde{\mathtt{S}}}(\mathcal{O}_{\tilde\wp})\\ &\xrightarrow{N_{E_{\tilde{\mathtt{S}}}/\mathbb{Q}}} T_{E,\tilde{\mathtt{S}}}(\mathbb{A}^\infty)/T_{E,\tilde{\mathtt{S}}}(\mathbb{Q})T_{E,\tilde{\mathtt{S}}}(\mathbb{Z}_p)\xrightarrow{\cong}\mathbb{A}_E^{\infty,p,\times}/\mathcal{O}_{E,(p)}^{\times,\mathrm{cl}}.\end{aligned}$$

For later use, we record the notation on the action of $\sigma_{\mathfrak{p}}$ for $\mathfrak{p}\in\Sigma_p$ on $\Sigma_{E,\infty}$: for $\tilde\tau\in\Sigma_{E,\infty}$, we put

$$\sigma_{\mathfrak{p}}\tilde\tau=\begin{cases}\sigma\circ\tilde\tau & \text{if } \tilde\tau\in\Sigma_{E,\infty/\mathfrak{p}},\\ \tilde\tau & \text{if } \tilde\tau\notin\Sigma_{E,\infty/\mathfrak{p}},\end{cases}\tag{5.4.1}$$

where $\Sigma_{E,\infty/\mathfrak{p}}$ denotes the lifts of places in $\Sigma_{\infty/\mathfrak{p}}$. Note that this $\sigma_{\mathfrak{p}}$ action is compatible with the one on Σ_∞ given in Subsection 4.6. Let $\sigma_{\mathfrak{p}}\tilde{\mathtt{S}}=(\mathtt{S}_p\cup\sigma_{\mathfrak{p}}\mathtt{S}_\infty,\sigma_{\mathfrak{p}}\tilde{\mathtt{S}}_\infty)$ with $\tilde{\mathtt{S}}_\infty$ the image of $\sigma_{\mathfrak{p}}\tilde{\mathtt{S}}_\infty$ under $\sigma_{\mathfrak{p}}$. The product $\sigma=\prod_{\mathfrak{p}\in\Sigma_p}\sigma_{\mathfrak{p}}$ is the usual Frobenius action, and we define $\sigma\tilde{\mathtt{S}}$ similarly.

5.5. Auxiliary Shimura varieties. — We also consider the product group $G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}}$ with the product Deligne homomorphism

$$\tilde{h}_{\tilde{\mathtt{S}}} = h_{\mathtt{S}} \times h_{E,\tilde{\mathtt{S}}} \colon \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow (G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}})(\mathbb{R}).$$

This gives rise to the product Shimura variety:

$$\mathbf{Sh}_{K_{\mathtt{S},p}\times K_{E,p}}(G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}}) = \mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}}) \times_{\mathcal{O}_\wp} \mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}}).$$

Let $Z = \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m$ denote the center of $G_{\mathtt{S}}$. Put $G''_{\tilde{\mathtt{S}}} = G_{\mathtt{S}} \times_Z T_{E,\tilde{\mathtt{S}}}$ which is the quotient of $G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}}$ by Z embedded anti-diagonally as $z \mapsto (z, z^{-1})$. We consider the homomorphism $h''_{\tilde{\mathtt{S}}} : \mathbb{S}(\mathbb{R}) \to G''_{\tilde{\mathtt{S}}}(\mathbb{R})$ induced by $\tilde{h}_{\tilde{\mathtt{S}}}$. We will consider open compact subgroups $K''_{\mathtt{S}} \subset G''_{\tilde{\mathtt{S}}}(\mathbb{A}^\infty)$ of the form $K''^{p}_{\mathtt{S}} K''_{\mathtt{S},p}$, where $K''^{p}_{\mathtt{S}}$ is an open compact subgroup of $G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$ and $K''_{\mathtt{S},p}$ is the image of $K_{\mathtt{S},p} \times K_{E,p}$ under the natural projection $G_{\mathtt{S}}(\mathbb{Q}_p) \times T_{E,\tilde{\mathtt{S}}}(\mathbb{Q}_p) \to G''_{\tilde{\mathtt{S}}}(\mathbb{Q}_p)$. (Note here that the level structure at p only depends on $\mathtt{S}$ but not its lift $\tilde{\mathtt{S}}$; this is why we suppress the tilde from the notation.) For $K''^{p}_{\mathtt{S}} \subset G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$ sufficiently small, the corresponding Shimura variety admits a smooth integral model $\mathbf{Sh}_{K''_{\mathtt{S}}}(G''_{\tilde{\mathtt{S}}})$ over $\mathcal{O}_{\tilde{\wp}}$ [**TX13a**, Corollary 3.18]. Taking the limit for prime-to-p levels, we get $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}}) = \varprojlim_{K''^{p}_{\mathtt{S}}} \mathbf{Sh}_{K''^{p}_{\mathtt{S}} K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})$.

For $\mathfrak{p} \in \Sigma_p$, we use $S_{\mathfrak{p}}$ denote the Hecke action on $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})$ given by multiplication by $(1, \varpi_{\mathfrak{p}}^{-1})$, where $\varpi_{\mathfrak{p}}$ is the uniformizer of $\mathcal{O}_{F,\mathfrak{p}}$ embedded in $(\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathfrak{p}})^\times \subset T_{E,\tilde{\mathtt{S}}}(\mathbb{A}^\infty)$.

Let $\alpha : G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}} \to G''_{\tilde{\mathtt{S}}}$ denote the natural projection. The homomorphisms of algebraic groups induces a diagram of (projective systems of) Shimura varieties:

(5.5.1)
$$\begin{array}{ccccc} \mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}}) & \xleftarrow{\mathrm{pr}_1} & \mathbf{Sh}_{K_{\mathtt{S},p}\times K_{E,p}}(G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}}) & \xrightarrow{\boldsymbol{\alpha}} & \mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}}) \\ & & \big\downarrow{\scriptstyle \mathrm{pr}_2} & & \\ & & \mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}}) & & \end{array}$$

Note that the corresponding Deligne homomorphism is compatible for all morphisms of the algebraic groups.

5.6. Automorphic sheaves on Shimura varieties. — Let L be the number field as in Subsection 3.1. Note that $G_{\mathtt{S}} \times_{\mathbb{Q}} L = \prod_{\tau \in \Sigma_\infty} \mathrm{GL}_{2,L}$, where $F^\times$ acts on the τ-component through τ. Given a multiweight $(\underline{k}, w)$, we consider the following algebraic representation of $G_{\mathtt{S}} \times_{\mathbb{Q}} L$:

$$\rho^{(\underline{k},w)}_{\mathtt{S}} = \bigotimes_{\tau \in \Sigma_\infty} \rho^{(k_\tau,w)}_{\tau} \circ \check{\mathrm{pr}}_\tau \quad \text{with} \quad \rho^{(k_\tau,w)}_{\tau} = \mathrm{Sym}^{k_\tau - 2} \otimes \det^{\frac{w-k_\tau}{2}},$$

where $\check{\mathrm{pr}}_\tau$ is the contragradient of the natural projection to the τ-component of $G_{\mathtt{S}} \times_{\mathbb{Q}} L$. The representation $\rho^{(\underline{k},w)}$ factors through the quotient group $G^c_{\mathtt{S}} = G_{\mathtt{S}} / \mathrm{Ker}(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m \to \mathbb{G}_m)$. By [**Mil90a**, Ch. III, § 7], the representation $\rho^{(\underline{k},w)}$ gives rise to an $\overline{\mathbb{Q}}_l$-lisse sheaf $\mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l}$ on $\mathrm{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$ which naturally extends to its integral model $\mathbf{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$.

For the l-adic local system on $\mathbf{Sh}_{K''_{\mathtt{S}}}(G''_{\tilde{\mathtt{S}}})$, we need to fix a section of the natural map $\Sigma_{E,\infty} \to \Sigma_\infty$, that is to fix a lift $\tilde{\tau}$ for each $\tau \in \Sigma_\infty$; use $\widetilde{\Sigma} = \widetilde{\Sigma}_\infty$ to denote the image of the section. Till the end of this subsection, we use $\tilde{\tau}$ to denote this chosen lift of τ. Consider the injection

$$G''_{\tilde{\mathtt{S}}} \times_{\mathbb{Q}} L = (B^\times_{\mathtt{S}} \times_{F^\times} E^\times) \times_{\mathbb{Q}} L \hookrightarrow \mathrm{Res}_{E/\mathbb{Q}}(B_{\mathtt{S}} \otimes_F E)^\times \times_{\mathbb{Q}} L \cong \prod_{\tau \in \Sigma_\infty} \mathrm{GL}_{2,L,\tilde{\tau}} \times \mathrm{GL}_{2,L,\tilde{\tau}^c},$$

where $E^\times$ acts on $\mathrm{GL}_{2,L,\tilde{\tau}}$ (resp. $\mathrm{GL}_{2,L,\tilde{\tau}^c}$) through $\tilde{\tau}$ (resp. complex conjugate of $\tilde{\tau}$). For a multiweight $(\underline{k}, w)$, we consider the following representation of $G''_{\tilde{\mathtt{S}}} \times_{\mathbb{Q}} L$:

$$\rho''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\widetilde{\Sigma}} = \bigotimes_{\tau \in \Sigma_\infty} \rho^{(k_\tau,w)}_{\tau} \circ \check{\mathrm{pr}}_{\tilde{\tau}} \quad \text{with} \quad \rho^{(k_\tau,w)}_{\tau} = \mathrm{Sym}^{k_\tau - 2} \otimes \det^{\frac{w-k_\tau}{2}},$$

where $\check{\mathrm{pr}}_{\tilde{\tau}}$ is the contragradient of the natural projection to the $\tilde{\tau}$-component of $G''_{\tilde{\mathtt{S}}} \times_{\mathbb{Q}} L \hookrightarrow \mathrm{Res}_{E/\mathbb{Q}} D^{\times}_{\mathtt{S}} \times_{\mathbb{Q}} L$. By [**Mil90a**, Ch. III, § 7], the representation $\rho''^{(\underline{k},w)}_{\mathtt{S},\widetilde{\Sigma}}$ gives rise to an $\overline{\mathbb{Q}}_l$-lisse sheaf $\mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\widetilde{\Sigma},l}$ on $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})$.

We also consider the following one-dimensional representation of $\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m \times_{\mathbb{Q}} L \cong \prod_{\tau \in \Sigma_\infty} \mathbb{G}_{m,\tilde{\tau}} \times \mathbb{G}_{m,\tilde{\tau}^c}$:

$$\rho^w_{E,\tilde{\Sigma}} = \bigoplus_{\tau \in \Sigma_\infty} x^{2-w} \circ \mathrm{pr}_{E,\tilde{\tau}},$$

where $\mathrm{pr}_{E,\tilde{\tau}}$ is the projection to the $\tilde{\tau}$-component and x^{2-w} is the character of $\mathbb{C}^\times$ given by raising to $(2-w)$th power. These representations give rise to a lisse $\overline{\mathbb{Q}}_l$-sheaf $\mathscr{L}^w_{E,\widetilde{\Sigma},l}$ on $\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})$.

We have an isomorphism of representations of $G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}}$

$$\rho''^{(\underline{k},w)}_{\mathtt{S},\widetilde{\Sigma}} \circ \alpha \cong \big(\rho^{(\underline{k},w)}_{\mathtt{S}} \circ \mathrm{pr}_1\big) \otimes \big(\rho^w_{E,\widetilde{\Sigma}} \circ \mathrm{pr}_2\big) \quad \text{for any } \widetilde{\Sigma},$$

and hence an isomorphism of $\overline{\mathbb{Q}}_l$-étale sheaves on $\mathbf{Sh}_{K_{\mathtt{S},p} \times K_{E,p}}(G_{\mathtt{S}} \times T_{E,\tilde{\mathtt{S}}})$:

$$\widetilde{\mathscr{L}}^{(\underline{k},w)}_{\tilde{\mathtt{S}},\widetilde{\Sigma},l} := \boldsymbol{\alpha}^* \mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\widetilde{\Sigma},l} \cong \mathrm{pr}_1^* \mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l} \otimes \mathrm{pr}_2^* \mathscr{L}^w_{E,\widetilde{\Sigma},l} \quad \text{for any } \widetilde{\Sigma}. \tag{5.6.1}$$

Remark 5.7. — The $\overline{\mathbb{Q}}_l$-étale sheaves $\mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l}$, $\mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\widetilde{\Sigma},l}$, $\mathscr{L}^w_{E,\widetilde{\Sigma},l}$ and $\widetilde{\mathscr{L}}^{(\underline{k},w)}_{\tilde{\mathtt{S}},\widetilde{\Sigma},l}$ are base change of $L_{\mathfrak{l}}$-sheaves on the corresponding Shimura varieties, where $\mathfrak{l}$ is the l-adic place of L determined by the isomorphism $\iota_l : \mathbb{C} \simeq \overline{\mathbb{Q}}_l$. For the computation of cohomology in terms of automorphic forms, it is more convenient to work with $\overline{\mathbb{Q}}_l$-coefficients.

5.8. Family of Abelian varieties. — We summarize the basic properties of certain abelian varieties over $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})$ constructed in [**TX13a**].

1. [**TX13a**, §3.20] There is a natural family of abelian varieties $\mathbf{A}'' = \mathbf{A}''_{\tilde{\mathtt{S}}}$ of dimension $4g$ over $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})$ equipped with a natural action of $\mathrm{M}_2(\mathcal{O}_E)$ and satisfies some Kottwitz's determinant condition. There is a (commutative) equivariant action of $G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$ on $\mathbf{A}''$ so that for sufficiently small $K''^p_{\mathtt{S}} \subset G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$, the abelian scheme $\mathbf{A}''$ descends to $\mathbf{Sh}_{K''^p_{\mathtt{S}} K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})$.
2. [**TX13a**, §3.21] Let $a'' : \mathbf{A}'' \to \mathbf{Sh}_{K''_{\mathtt{S}}}(G''_{\tilde{\mathtt{S}}})$ be the structural morphism. The direct image $R^1 a''_*(\overline{\mathbb{Q}}_l)$ has a canonical decomposition:

$$R^1 a''_*(\overline{\mathbb{Q}}_l) = \bigoplus_{\tau \in \Sigma_\infty} \big(R^1 a''_*(\overline{\mathbb{Q}}_l)_{\tilde{\tau}} \oplus R^1 a''_*(\overline{\mathbb{Q}}_l)_{\tilde{\tau}^c}\big)$$

   where $R^1 a''_*(\overline{\mathbb{Q}}_l)_{\tilde{\tau}}$ (resp. $R^1 a''_*(\overline{\mathbb{Q}}_l)_{\tilde{\tau}^c}$) is the direct summand where $\mathcal{O}_E$ acts via $\tilde{\tau}$ (resp. via $\tilde{\tau}^c$). Let $\mathfrak{e} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O}_E)$. We put $R^1 a''_*(L_{\mathfrak{l}})^{\circ}_{\tilde{\tau}} = \mathfrak{e} R^1 a''_*(\overline{\mathbb{Q}}_l)_{\tilde{\tau}}$ for $\tau \in \Sigma_\infty$. This is a $\overline{\mathbb{Q}}_l$-lisse sheaf over $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})$ of rank 2. For a multiweight $(\underline{k}, w)$, we have an isomorphism of $\overline{\mathbb{Q}}_l$-lisse sheaves:

$$\mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l} \cong \bigotimes_{\tau \in \Sigma_\infty} \left(\mathrm{Sym}^{k_\tau - 2} R^1 a''_*(\overline{\mathbb{Q}}_l)^{\circ}_{\tilde{\tau}} \otimes (\wedge^2 R^1 a''_*(\overline{\mathbb{Q}}_l)^{\circ}_{\tilde{\tau}})^{\frac{w-k_\tau}{2}}\right).$$

3. [**TX13a**, Proposition 3.23] For each $\mathfrak{p} \in \Sigma_p$, we have a natural $G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$-equivariant *twisted partial Frobenius morphism* and an qausi-isogeny of family of abelian varieties.

$$\mathfrak{F}''_{\mathfrak{p}^2} : \mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})_{k_{\tilde{\wp}}} \longrightarrow \mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\sigma^2_{\mathfrak{p}}\tilde{\mathtt{S}}})_{k_{\tilde{\wp}}} \quad \text{and} \quad \eta''_{\mathfrak{p}^2} : \mathbf{A}''_{\tilde{\mathtt{S}},k_{\tilde{\wp}}} \longrightarrow \mathfrak{F}''^*_{\mathfrak{p}^2}(\mathbf{A}''_{\sigma^2_{\mathfrak{p}}\tilde{\mathtt{S}},k_{\tilde{\wp}}}). \tag{5.8.1}$$

   This induces a natural $G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$-equivariant homomorphism of étale cohomology groups:

$$\Phi_{\mathfrak{p}^2} : H^*_{\mathrm{et}}\big(\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\sigma^2_{\mathfrak{p}}\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}''^{(\underline{k},w)}_{\sigma^2_{\mathfrak{p}}\tilde{\mathtt{S}},\tilde{\Sigma},l}\big) \longrightarrow H^*_{\mathrm{et}}\big(\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l}\big),$$

where $\sigma_{\mathfrak{p}}$ is as defined at the end of Subsection 5.4. Moreover, we have an equality of morphisms

$$\prod_{\mathfrak{p}\in\Sigma_p}\Phi_{\mathfrak{p}^2} = S_p^{-1}\circ F^2\colon H^*_{\mathrm{et}}\big(\mathbf{Sh}_{K''_p}(G''_{\sigma^2\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}''^{(\underline{k},w)}_{\sigma^2\tilde{\mathtt{S}},\tilde{\Sigma},l}\big) \longrightarrow H^*_{\mathrm{et}}\big(\mathbf{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l}\big),$$

where F^2 is the relative p^2-Frobenius, σ is as defined at the end of Subsection 5.4, and S_p is the Hecke action given by the central element $\underline{p}^{-1}\in\mathbb{A}_E^{\infty,\times}\subset G''(\mathbb{A}^\infty)$. Here, $\underline{p}$ is the idèle which is p at all p-adic places of E and 1 otherwise.

4. When $\mathtt{S}=\emptyset$, let $\mathcal{A}$ denote the universal abelian variety over the Hilbert modular variety $\mathbf{Sh}_{K_p}(G)$. One has an isomorphism of abelian schemes over $\mathbf{Sh}_{K_{\mathtt{S},p}\times K_{E,p}}(G_{\mathtt{S}}\times T_{E,\tilde{\mathtt{S}}})$ [**TX13a**, Corollary 3.26]:

$$\boldsymbol{\alpha}^*\mathbf{A}''\simeq(\mathrm{pr}_1^*\mathcal{A}\otimes_{\mathcal{O}_F}\mathcal{O}_E)^{\oplus 2}, \tag{5.8.2}$$

compatible with $\mathrm{M}_2(\mathcal{O}_E)$-actions, where $\boldsymbol{\alpha}$ is defined in (5.5.1). Moreover, the morphism $\boldsymbol{\alpha}$ and the isomorphism (5.8.2) are compatible with the action of twisted partial Frobenius [**TX13a**, Cor 3.28].

5. [**TX13a**, 4.7, 4.8, 4.10] Let k_0 be a finite extension of $\mathbb{F}_p$ containing all residue fields of E of characteristic p. The special fiber $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0}$ admits a GO-stratification, that is a *smooth* $G''_{\emptyset}(\mathbb{A}^{\infty,p})$-stable subvariety $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0,\mathtt{T}}$ for each subset $\mathtt{T}\subseteq\Sigma_\infty$ (given by the vanishing locus of certain variants of partial Hasse invariants). We refer to *loc. cit.* for the precise definition. The twisted partial Frobenius morphism $\mathfrak{F}''_{\mathfrak{p}^2}$ in (5.8.1) takes the subvariety $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})_{k_0,\mathtt{T}}$ to $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\sigma_{\mathfrak{p}}^2\tilde{\mathtt{S}}})_{k_0,\sigma_{\mathfrak{p}}^2\mathtt{T}}$. Here, we view $K''_{\mathtt{S}}$ also as a subgroup of $G''_{\sigma_{\mathfrak{p}}^2\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$ via a fixed isomorphism $G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})\simeq G''_{\sigma_{\mathfrak{p}}^2\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$.

The GO-stratification on $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0}$ is compatible with the Goren-Oort stratification on the Hilbert modular variety $\mathbf{Sh}_{K_p}(G)_{k_0}$ recalled in Subsection 3.2 in the sense that

$$\boldsymbol{\alpha}^{-1}(\mathbf{Sh}_{K''_p}(G''_{\emptyset})_{k_0,\mathtt{T}})\cong\mathrm{pr}_1^{-1}(\mathbf{Sh}_{K_p}(G)_{k_0,\mathtt{T}})\quad\text{for all }\mathtt{T}\subseteq\Sigma_\infty. \tag{5.8.3}$$

5.9. Tensorial induced representation. — We recall the definition of tensorial induced representation. Let G be a group and H a subgroup of finite index. Let (ρ,V) be a finite dimensional representation of H. Let $\Sigma\subseteq G/H$ be a finite subset. Consider the left action of G on the set of left cosets G/H. Let H' be the subgroup of G that fixes the subset Σ of G/H. Fix representatives $s_1,\dots,s_n$ of G/H and we may assume that $\Sigma=\{s_1H,\dots,s_rH\}$ for some r.

The *tensorial induced representation*, denoted by $\otimes_\Sigma\text{-}\mathrm{Ind}_H^G V$, is defined to be $\otimes_{i=1}^r V_i$, where V_i is a copy of V. The action of H' is given as follows: for a given $h'\in H'$ and for each $i\in\{1,\dots,r\}$, there exists a unique $j(i)\in\{1,\dots,r\}$, we have $h's_{j(i)}\in s_iH$; then we define

$$h'(v_1\otimes\cdots\otimes v_r)=\rho(s_1^{-1}h's_{j(1)})(v_{j(1)})\otimes\cdots\otimes\rho(s_r^{-1}h's_{j(r)})(v_{j(r)}).$$

One can easily check that this definition of $\otimes_\Sigma\text{-}\mathrm{Ind}_H^G V$ does not depend on the choice of coset representatives.

We will need the following two instances of the construction above: (1) $G=\mathrm{Gal}_{\mathbb{Q}}$, $H=\mathrm{Gal}_F$, and $\Sigma=\Sigma_\infty-\mathtt{S}_\infty\subseteq\Sigma_\infty\simeq G/H$. Then the subgroup H' is $\mathrm{Gal}_{F_{\mathtt{S}}}$; (2) $G=\mathrm{Gal}_{\mathbb{Q}}$, $H=\mathrm{Gal}_E$, and $\Sigma=\tilde{\Sigma}-\tilde{\mathtt{S}}_\infty$ (see Subsection 5.8), viewed as a subset of $\Sigma_{E,\infty}\simeq G/H$. Then the subgroup H' is $\mathrm{Gal}_{E_{\mathtt{S}}}$.

5.10. Automorphic representations of $\mathrm{GL}_{2,F}$. — For a multiweight $(\underline{k},w)$, let $\mathscr{A}_{(\underline{k},w)}$ denote the set of irreducible cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ whose archimedean component π_τ for each $\tau\in\Sigma_\infty$ is a discrete series of weight $k_\tau-2$ with central character $x\mapsto x^{w-2}$. It follows that the central character $\omega_\pi:\mathbb{A}_F^\times/F^\times\to\mathbb{C}^\times$ for such π can be written as $\omega_\pi=\varepsilon_\pi|\cdot|_F^{w-2}$, with ε_π a finite character Hecke character trivial on $(F\otimes\mathbb{R})^\times$.

Given $\pi \in \mathscr{A}_{(\underline{k},w)}$, the finite part π^∞ of π can be defined over a number field (viewed as a subfield of $\mathbb{C}$). For an even subset $\mathtt{S} \subseteq \Sigma$, we use $\pi_{\mathtt{S}}$ to denote the Jacquet-Langlands transfer of π to an automorphic representation over $B_{\mathtt{S}}^\times(\mathbb{A}_F)$; it is zero if π does not transfer.

Thanks to the work of many people [**Ca86b, Ta89, BR93**], we can associate to π a 2-dimensional Galois representation $\rho_{\pi,l} : \mathrm{Gal}_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$. We normalize $\rho_{\pi,l}$ so that $\det(\rho_{\pi,l}) = \varepsilon_\pi^{-1} \cdot \chi_{\mathrm{cyc}}^{1-w}$, where χ_{cyc} is the l-adic cyclotomic character and ε_π is the finite character above, viewed as a Galois representation with values in $\overline{\mathbb{Q}}_l$ via class field theory and the isomorphism $\iota_l : \mathbb{C} \cong \overline{\mathbb{Q}}_l$. If π^∞ is defined over a number field $L \subseteq \mathbb{C}$, then $\rho_{\pi,l}$ is rational over $L_{\mathfrak{l}}$, where $\mathfrak{l}$ denotes the l-adic place of L determined by ι_l.

When $k_\tau = 2$ for all τ, $w \geq 2$ is an even integer. We denote by $\mathscr{B}_w$ the set of irreducible automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ which factor as

$$\mathrm{GL}_2(\mathbb{A}_F) \xrightarrow{\det} \mathbb{A}_F^\times/F^\times \xrightarrow{\chi} \mathbb{C}^\times,$$

where χ is an algebraic Hecke character whose restriction to F_τ^+ for each $\tau \in \Sigma_\infty$ is $x \mapsto x^{\frac{w}{2}-1}$. With the fixed isomorphism $\iota_l : \mathbb{C} \cong \overline{\mathbb{Q}}_l$, we define an l-adic character on $\mathbb{A}_F^\times$ given by

$$\chi_l : x \mapsto \big(\chi(x) \cdot N_{F/\mathbb{Q}}(x_\infty)^{1-\frac{w}{2}}\big) \cdot N_{F/\mathbb{Q}}(x_l)^{\frac{w}{2}-1} \in \overline{\mathbb{Q}}_l^\times$$

where $x_\infty \in (F \otimes \mathbb{R})^\times$ (resp. $x_l \in (F \otimes \mathbb{Q}_l)^\times$) denote the archimedean components (resp. l-components of x). Note also that χ_l is trivial on $F^\times$, and hence by class field theory gives rise to a l-adic Galois character on Gal_F. We put $\rho_{\pi,l} = \chi_l^{-1}$. Also note that the map $x \mapsto \chi(x) N_{F/\mathbb{Q}}(x_\infty)^{1-\frac{w}{2}}$ on $\mathbb{A}_F^\times$ has values in a number field. We may choose a number field $L \subseteq \mathbb{C}$ large enough so that $\rho_{\pi,l}$ is rational over $L_{\mathfrak{l}}$. Given $\pi \in \mathscr{B}_w$ and an even subset $\mathtt{S}$, we denote by $\pi_{\mathtt{S}}$ the one-dimensional automorphic representation of $G_{\mathtt{S}}$ that factors as $G_{\mathtt{S}}(\mathbb{A}) \xrightarrow{\nu_{\mathtt{S}}} \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, where $\nu_{\mathtt{S}}$ is the reduced norm map.

5.11. Cohomology of $\mathbf{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})$. — Let $\mathtt{S}$ be an even subset of places of F satisfying Hypothesis 5.2. Let $K_{\mathtt{S}} \subset G_{\mathtt{S}}(\mathbb{A}^\infty)$ be an open compact subgroup. We fix an isomorphism $G_{\mathtt{S}}(\mathbb{A}^{\infty,p}) \simeq \mathrm{GL}_2(\mathbb{A}^{\infty,p})$. For a field M of characteristic 0, let $\mathscr{H}(K_{\mathtt{S}}^p, M)$ denote the prime-to-p Hecke algebra $M[K_{\mathtt{S}}^p\backslash \mathrm{GL}_2(\mathbb{A}^\infty)/K_{\mathtt{S}}^p]$.

For each $\pi \in \mathscr{A}_{(\underline{k},w)}$ or $\pi \in \mathscr{B}_w$, let $(\pi_{\mathtt{S}}^\infty)^{K_{\mathtt{S}}} = (\pi_{\mathtt{S}}^{\infty,p})^{K_{\mathtt{S}}^p} \otimes (\pi_{\mathtt{S},p})^{K_{\mathtt{S},p}}$ be the $K_{\mathtt{S}}$-invariant subspace of $\pi_{\mathtt{S}}^\infty$. We consider it as a $\mathscr{H}(K_{\mathtt{S}}^p, \mathbb{C})$-module with the natural Hecke action of $\mathscr{H}(K_{\mathtt{S}}^p, \mathbb{C})$ on $(\pi_{\mathtt{S}}^{\infty,p})^{K_{\mathtt{S}}^p}$ and trivial action on $(\pi_{\mathtt{S},p})^{K_{\mathtt{S},p}}$. The following result is well-known.

Theorem 5.12. — *In the Grothendieck group of modules over $\mathscr{H}(K_{\mathtt{S}}, \overline{\mathbb{Q}}_l)[\mathrm{Gal}_{F_{\mathtt{S}}}]$, we have an equality*

$$\begin{aligned}\big[H^\star_{c,\mathrm{et}}(\mathrm{Sh}_{K_{\mathtt{S}}}(G_{\mathtt{S}})_{\overline{\mathbb{Q}}}, \mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l})\big] &= (-1)^{g-\#\mathtt{S}_\infty} \sum_{\pi \in \mathscr{A}_{(\underline{k},w)}} \big[(\pi_{\mathtt{S}}^\infty)^{K_{\mathtt{S}}} \otimes \rho_{\pi,l}^{\mathtt{S}}\big] \\ &+ \delta_{\underline{k},2} \sum_{\pi \in \mathscr{B}_w} \big[(\pi_{\mathtt{S}}^\infty)^{K_{\mathtt{S}}} \otimes \rho_{\pi,l}^{\mathtt{S}}\big] \otimes \Big(\big[(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l(-1))^{\otimes(\Sigma_\infty - \mathtt{S}_\infty)}\big] - \delta_{\mathtt{S},\emptyset}\big[\overline{\mathbb{Q}}_l\big]\Big)\end{aligned}$$

Here, for each π in $\mathscr{A}_{(\underline{k},w)}$ or $\mathscr{B}_w$, we put $\rho_{\pi,l}^{\mathtt{S}} := \bigotimes_{\Sigma_\infty - \mathtt{S}_\infty}\text{-}\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}}(\rho_{\pi,l})$ where $\rho_{\pi,l}$ denotes the l-adic representation of Gal_F defined above, $\delta_{\underline{k},2}$ equals 1 if $\underline{k} = (2, \dots, 2)$ and 0 otherwise, and $\delta_{\mathtt{S},\emptyset} = 1$ if $\mathtt{S} = \emptyset$ and 0 otherwise.

Proof. — For $\mathtt{S} \neq \emptyset$, this is proved in [**BL84**, §3.2]. When $\mathtt{S} = \emptyset$, the contributions from cuspidal representations and one-dimensional representations are computed in the same way as above in *loc. cit.*; the subtraction by $\overline{\mathbb{Q}}_l$ when $\underline{k} = (2, \dots, 2)$ comes from the fact that $H_c^0(\mathrm{Sh}_K(G)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_l) = 0$. We explain now why there is no contributions from Eisenstein series in the Grothendieck group. Actually, according to [**MSYZ**], the Eisenstein spectra appear in H_c^i only when $\underline{k}$ is of parallel weight. In that case, each possible Eisenstein series will only appear in H_c^i with multiplicity $\binom{g-1}{i}$ for $1 \leq i \leq g$, and none in H_c^i with $i = 0$ or $i \geq g+1$. Hence, their contributions cancel out when passing to the Grothendieck group. $\square$

5.13. Cohomology of $\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})$. — Let $w \in \mathbb{Z}$ and $\tilde{\Sigma} = \tilde{\Sigma}_{E,\infty}$ be as in Subsection 5.6. Let $\mathscr{A}^w_{E,\tilde{\Sigma}}$ denote the set of Hecke characters χ of $\mathbb{A}_E^\times/E^\times$ such that $\chi|_{E_{\tilde{\tau}}^\times} : x \mapsto x^{w-2}$ for all $\tau \in \Sigma_\infty$ and χ is unramified at places above p. Here, the isomorphism $E_{\tilde{\tau}} = E \otimes_{F,\tau} \mathbb{R} \xrightarrow{\sim} \mathbb{C}$ is defined with the embedding $\tilde{\tau} : E \hookrightarrow \mathbb{C}$.

We fix an isomorphism $\iota_l : \mathbb{C} \cong \overline{\mathbb{Q}}_l$ as before. Then we can identify each $\tilde{\tau} \in \Sigma_{E,\infty}$ with an embedding $\tilde{\tau}_l$ of E into $\overline{\mathbb{Q}}_l$. Define an l-adic character on $\mathbb{A}_E^\times$ associated to χ:

$$\chi_l \colon x \mapsto \big(\chi(x) \cdot \prod_{\tilde{\tau} \in \tilde{\Sigma}} \tilde{\tau}(x)^{2-w}\big) \cdot \prod_{\tilde{\tau} \in \tilde{\Sigma}} \tilde{\tau}_l(x)^{w-2} \in \overline{\mathbb{Q}}_l^\times.$$

This character factors through $E^\times \backslash \mathbb{A}_E^\times / E_{\mathbb{R}}^\times$ and hence induces a Galois representation $\chi_l \colon \mathrm{Gal}_E \to \overline{\mathbb{Q}}_l^\times$ via class field theory. We put $\rho_{\chi,l} = \chi_l^{-1}$.

Given $\mathtt{S}$ and $\tilde{\mathtt{S}}_\infty$ as in Subsection 5.8, we view $\tilde{\mathtt{S}}_\infty$ as a subset of $\Sigma_{E,\infty} \cong \mathrm{Gal}_{\mathbb{Q}} / \mathrm{Gal}_E$, where $\mathrm{Gal}_{\mathbb{Q}}$ acts on the left by postcomposition. The construction in Subsection 5.9 gives rise to a representation of $\mathrm{Gal}_{E_{\tilde{\mathtt{S}}}}$

$$\rho_{\chi,\tilde{\mathtt{S}},l} := \bigotimes_{\tilde{\mathtt{S}}_\infty} \text{-}\,\mathrm{Ind}_{\mathrm{Gal}_E}^{\mathrm{Gal}_{\mathbb{Q}}} \rho_{\chi,l}.$$

Lemma 5.14. — *We have an $\overline{\mathbb{Q}}_l[\mathbb{A}_E^{\infty,\times}] \times \mathrm{Gal}_{k_{\tilde{\wp}}}$-equivariant isomorphism:*

$$H^0_{\mathrm{et}}\big(\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}^w_{\tilde{\mathtt{S}},E,l}\big) \simeq \bigoplus_{\chi \in \mathscr{A}^w_{E,\tilde{\Sigma}}} \chi \otimes \rho_{\chi,\tilde{\mathtt{S}},l}|_{\mathrm{Gal}_{k_{\tilde{\wp}}}}, \tag{5.14.1}$$

where $\mathbb{A}_E^{\infty,\times}$ acts on the right hand side via (the finite part) of χ, and $\mathrm{Gal}_{k_{\tilde{\wp}}}$ acts via $\rho_{\chi,\tilde{\mathtt{S}},l}$.

Proof. — According to Deligne's definition of Shimura varieties for tori, the action of $\mathrm{Frob}_{\wp}$ is the same as the Hecke action of the element $\mathfrak{Rec}_{E,\tilde{\mathtt{S}}}(\varpi_{\tilde{\mathtt{S}}})$, where the map $\mathfrak{Rec}_{E,\tilde{\mathtt{S}}}$ is the reciprocity map defined in Subsection 5.4. It follows that the Galois action on $H^0_{\mathrm{et}}\big(\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}^w_{\tilde{\mathtt{S}},E,l}\big)$ is as described. □

The following lemma will be used later.

Lemma 5.15. — *Keep the notation as above. Put $d_{\tilde{\wp}} = [k_{\tilde{\wp}} : \mathbb{F}_p]$. Let $d_{\mathfrak{q}}$ denote the inertia degree of a p-adic place $\mathfrak{q} \in \Sigma_{E,p}$ in $E/\mathbb{Q}$. Let $\tilde{\mathtt{S}}_{\infty,\mathfrak{q}}$ denote the set of places in $\tilde{\mathtt{S}}_\infty$ inducing the place $\mathfrak{q}$ of E via $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$. Let $\mathrm{Frob}_{\tilde{\wp}}$ denote the geometric Frobenius for $k_{\tilde{\wp}}$. Then $\rho_{\chi,\tilde{\mathtt{S}},l}(\mathrm{Frob}_{\tilde{\wp}}) = \prod_{\mathfrak{q} \in \Sigma_{E,p}} \chi_l(\varpi_{\mathfrak{q}})^{-d_{\tilde{\wp}} \# \tilde{\mathtt{S}}_{\infty/\mathfrak{q}}/d_{\mathfrak{q}}}$.*

Proof. — This is a straightforward calculation. For each $\mathfrak{q} \in \Sigma_{E,p}$, let $\mathrm{Frob}_{\mathfrak{q}}$ denote a geometric Frobenius of Gal_E at $\mathfrak{q}$. Then we have

$$\begin{aligned}\rho_{\chi,\tilde{\mathtt{S}},l}(\mathrm{Frob}_{\tilde{\wp}}) &= \bigotimes_{\mathfrak{q} \in \Sigma_{E,p}} \Big(\bigotimes_{\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}} \text{-}\,\mathrm{Ind}_{\mathrm{Gal}_{E_{\mathfrak{q}}}}^{\mathrm{Gal}_{\mathbb{Q}_p}} \rho_{\chi,l}|_{\mathrm{Gal}_{E_{\mathfrak{q}}}}\Big)(\mathrm{Frob}_{\tilde{\wp}}) \\ &= \prod_{\mathfrak{q} \in \Sigma_{E,p}} \rho_{\chi,l}(\mathrm{Frob}_{\mathfrak{q}})^{d_{\tilde{\wp}} \# \tilde{\mathtt{S}}_{\infty/\mathfrak{q}}/d_{\mathfrak{q}}} = \prod_{\mathfrak{q} \in \Sigma_{E,p}} \chi_l(\varpi_{\mathfrak{q}})^{-d_{\tilde{\wp}} \# \tilde{\mathtt{S}}_{\infty/\mathfrak{q}}/d_{\mathfrak{q}}}.\end{aligned}$$

□

When relating the étale cohomology of $\mathbf{Sh}_{K''_{\tilde{\mathtt{S}},p}}(G''_{\tilde{\mathtt{S}}})$ and $\mathbf{Sh}_{K_p}(G)$, we need the following.

Proposition 5.16. — *Let $\chi \in \mathscr{A}^w_{E,\tilde{\Sigma}}$, and χ_F be its restriction to $\mathbb{A}_F^{\infty,\times}$. Then the following hold:*

1. *We have a natural $G_{\mathtt{S}}(\mathbb{A}^{\infty,p}) \times \mathrm{Gal}_{E_{\tilde{\mathtt{S}}}}$-equivariant isomorphism*

$$H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}})_{\overline{\mathbb{F}}_p}, \mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l})^{\mathbb{A}_F^{\infty,\times} = \chi_F} \otimes \rho_{\chi,\tilde{\mathtt{S}},l} \cong H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_{\tilde{\mathtt{S}},p}}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l})^{\mathbb{A}_E^{\infty,\times} = \chi}, \tag{5.16.1}$$

where the superscripts mean to take the subspaces where the Hecke actions are given as described.

2. *When* $\mathtt{S}=\emptyset$*, we have analogous* $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$*-equivariant isomorphisms for all* $\mathtt{T}\subseteq\Sigma_\infty$*:*

$$H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K_p}(G)_{\overline{\mathbb{F}}_p,\mathtt{T}},\mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l})^{\mathbb{A}_F^{\infty,\times}=\chi_F}\cong H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_p}(G''_\emptyset)_{\overline{\mathbb{F}}_p,\mathtt{T}},\mathscr{L}''^{(\underline{k},w)}_{\emptyset,\tilde{\Sigma},l})^{\mathbb{A}_E^{\infty,\times}=\chi}. \tag{5.16.2}$$

Moreover, (5.16.2) *is equivariant for the action of* $\Phi_{\mathfrak{p}^2}$ *on both sides (see Subsections 4.6 and 5.8(6) for the definition of the action).*

Proof. — (1) We first claim that the quotient $\alpha: G_{\mathtt{S}}\times T_{E,\tilde{\mathtt{S}}}\to G''_{\tilde{\mathtt{S}}}$ induces an isomorphism of Shimura varieties

$$\big(\mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}})\times\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})\big)/\mathbb{A}_F^{\infty,p,\times}\cong\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}}), \tag{5.16.3}$$

where $\mathbb{A}_F^{\infty,p,\times}$ acts by the anti-diagonal Hecke action. For this, it is enough to show that the product

$$\mathbb{A}_F^{\infty,p,\times}\cdot\big(G_{\mathtt{S}}(\mathbb{Q})_+^{(p),\mathrm{cl}}\times\mathcal{O}_{E,(p)}^{\times,\mathrm{cl}}\big)$$

is already closed in the $G_{\mathtt{S}}(\mathbb{A}^{\infty,p})\times\mathbb{A}_E^{\infty,p,\times}$, where the superscript means to take closure inside the corresponding adelic group, $\mathbb{A}_F^{\infty,p,\times}$ embeds in the product anti-diagonally, and $G_{\mathtt{S}}(\mathbb{Q})_+^{(p)}$ denote p-integral element of $G_{\mathtt{S}}(\mathbb{Q})$ with totally positive determinant. For this, we take an open compact subgroup $U^p_{\mathtt{S}}$ of $G_{\mathtt{S}}(\mathbb{A}^{\infty,p})$ and intersect the product above with $U^p_{\mathtt{S}}\times\widehat{\mathcal{O}}_E^{(p),\times}$; we are left to prove that the product

$$\widehat{\mathcal{O}}_F^{(p),\times}\cdot\big((G_{\mathtt{S}}(\mathbb{Q})_+^{(p)}\cap U^p_{\mathtt{S}})^{\mathrm{cl}}\times\mathcal{O}_E^{\times,\mathrm{cl}}\big)$$

is closed in $U^p_{\mathtt{S}}\times\widehat{\mathcal{O}}_E^{(p),\times}$. But Dirichlet's unit Theorem implies that $\mathcal{O}_F^\times$ is a finite index subgroup of $\mathcal{O}_E^\times$; it follows that the above product is a finite union of $\widehat{\mathcal{O}}_F^{(p),\times}\cdot(G_{\mathtt{S}}(\mathbb{Q})_+^{(p)}\cap U^p_{\mathtt{S}})^{\mathrm{cl}}\cdot$, which is obviously closed in $U^p_{\mathtt{S}}\times\widehat{\mathcal{O}}_E^{(p),\times}$. This proves the claim.

This claim in particular implies that for any $\overline{\mathbb{Q}}_l$-lisse sheaf $\mathscr{L}''$ on $\mathbf{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_{k_{\tilde{\wp}}}$, we have a natural isomorphism

$$H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p},\mathscr{L}'')\cong H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K_{\mathtt{S},p}\times K_{E,p}}(G_{\mathtt{S}}\times T_{E,\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p},\boldsymbol{\alpha}^*\mathscr{L}'')^{\text{anti-diag }\mathbb{A}_F^{\infty,p,\times}=1},$$

where the superscript means subspace where the anti-diagonal $\mathbb{A}_F^{\infty,p,\times}$ acts trivially.

Applying this to (5.6.1), and further taking the subspace where $\mathbb{A}_E^{\infty,p,\times}$ acts via χ (note that the restriction of a Hecke character to finite idèles away from p already determines its value at places above p), we get

$$\begin{aligned}
&H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_{\tilde{\mathtt{S}},p}}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p},\mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l})^{\mathbb{A}_E^{\infty,\times}=\chi}\\
\cong{}& H^\star_{c,\mathrm{et}}\big(\mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}})_{\overline{\mathbb{F}}_p}\times_{\overline{\mathbb{F}}_p}\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p},\boldsymbol{\alpha}^*\mathscr{L}^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l}\big)^{\mathbb{A}_F^{\infty,\times}\times\mathbb{A}_E^{\infty,\times}=\chi_F\times\chi}\\
\xrightarrow{\cong,(5.6.1)}{}& H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}})_{\overline{\mathbb{F}}_p},\mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l})^{\mathbb{A}_F^{\infty,\times}=\chi_F}\otimes H^\star_{\mathrm{et}}\big(\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p},\mathscr{L}^w_{E,\tilde{\Sigma},l}\big)^{\mathbb{A}_E^{\infty,\times}=\chi}\\
\xrightarrow{\cong,(5.14.1)}{}& H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}})_{\overline{\mathbb{F}}_p},\mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l})^{\mathbb{A}_F^{\infty,\times}=\chi_F}\otimes\rho_{\chi,\tilde{\mathtt{S}},l}.
\end{aligned}$$

This proves (5.16.1).

(2) Assume now $\mathtt{S}=\emptyset$. Consider the base change to k_0 of the isomorphism (5.16.3). Since the GO-stratification on $\mathbf{Sh}_{K_p}(G)_{k_0}$ and that on $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_\emptyset)_{k_0}$ are compatible (5.8.3), we have an isomorphism

$$\big(\mathbf{Sh}_{K_p}(G)_{k_0,\mathtt{T}}\times\mathbf{Sh}_{K_{E,p}}(T_{E,\emptyset})_{k_0}\big)/\mathbb{A}_F^{\infty,p,\times}\cong\mathbf{Sh}_{K''_{\emptyset,p}}(G''_\emptyset)_{k_0,\mathtt{T}},$$

for all subset $\mathtt{T}\subseteq\Sigma_\infty$. Then the same argument as above applies to the cohomology of $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_\emptyset)_{\overline{\mathbb{F}}_p,\mathtt{T}}$. This then proves (5.16.2). Here, note that the Galois representation $\rho_{\chi,\tilde{\mathtt{S}},l}$

is trivial, since the Deligne homomorphism $h_{E,\emptyset}$ is trivial. The compatiblility with twisted partial Frobenius follows from Subsection 5.8(5). □

Notation 5.17. — Let $\chi \in \mathscr{A}^w_{E,\tilde{\Sigma}}$ be a Hecke character, and put $\chi_F = \chi|_{\mathbb{A}_F^{\infty,\times}}$. We denote by $\mathscr{A}_{(\underline{k},w)}[\chi_F]$ the subset of cuspidal automorphic representations $\pi \in \mathscr{A}_{(\underline{k},w)}$ for which the central character $\omega_\pi = \chi_F$. When $\underline{k} = (2,\dots,2)$, $w \geq 2$ is an even integer. We also denote by $\mathscr{B}_w[\chi_F]$ be the set of one-dimensional automorphic representations π of $\mathrm{GL}_{2,F}$ such that $\omega_\pi = \chi_F$.

We remark that every Hecke character χ_F whose archimedean component is $x_\infty \mapsto N_{F/\mathbb{Q}}(x_\infty)$ extends to a Hecke character $\chi \in \mathscr{A}^w_{E,\tilde{\Sigma}}$. Indeed, we may first fix an arbitrary Hecke character χ', and then $\omega_0 = \chi'|^{-1}_{\mathbb{A}_F^\times} \cdot \chi_F$ is a *finite* character trivial on $(F \otimes \mathbb{R})^\times$. Since $\mathbb{A}_F^{\infty,\times}/F^\times$ injects into $\mathbb{A}_E^{\infty,\times}/E^\times$, we may always find a finite character χ_0 of $\mathbb{A}_E^{\infty,\times}/E^\times$ extending ω_0. Then $\chi'\chi_0$ is a Hecke character of $\mathbb{A}_E^\times/E^\times$ extending χ_F.

The following conjecture on the action of twisted partial Frobenius on the cohomology of Shimura varieties is well-known to the experts.

***Conjecture 5.18* (Partial Frobenius)**. — *For each $\mathfrak{p} \in \Sigma_p$, let $n_\mathfrak{p}$ be the smallest positive number n such that $\sigma_\mathfrak{p}^n \mathtt{S}_\infty = \mathtt{S}_\infty$. Let $d_\mathfrak{p}$ denote the inertia degree of $\mathfrak{p}$ in $F/\mathbb{Q}$. Assume that for any p-adic place $\mathfrak{p} \in \Sigma_p$ that splits into two primes $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ in E, we have $\#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}} = \#\tilde{\mathtt{S}}_{\infty/\bar{\mathfrak{q}}}$. Then, for any Hecke character $\chi \in \mathscr{A}^w_{E,\tilde{\Sigma}}$, we have the following equality in the Grothendieck group of (finite generated) modules over $\overline{\mathbb{Q}}_l[\mathrm{GL}_2(\mathbb{A}^{\infty,p})][(\Phi_{\mathfrak{p}^2})^{n_\mathfrak{p}}; \mathfrak{p} \in \Sigma_p]$.*

$$\begin{aligned}\big[H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l})^{\mathbb{A}_E^{\infty,\times}=\chi}\big] &= (-1)^{g-\#\mathtt{S}_\infty}\Big[\bigoplus_{\pi\in\mathscr{A}_{(\underline{k},w)}[\chi_F]} (\pi_{\mathtt{S}}^\infty)^{K_{\mathtt{S},p}} \otimes \tilde{\rho}^{\mathtt{S}}_{\pi,l}\Big] \\ &+ \delta_{\underline{k},2} \sum_{\pi\in\mathscr{B}_w[\chi_F]} \big[(\pi_{\mathtt{S}}^\infty)^{K_{\mathtt{S},p}} \otimes \tilde{\rho}^{\mathtt{S}}_{\pi,l}\big] \otimes \Big(\big[(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l(-1))^{\otimes(\Sigma_\infty - \mathtt{S}_\infty)}\big] - \delta_{\mathtt{S},\emptyset}[\overline{\mathbb{Q}}_l]\Big).\end{aligned} \tag{5.18.1}$$

Here, for each π in $\mathscr{A}_{(\underline{k},w)}[\chi_F]$ or $\mathscr{B}_w[\chi_F]$, we take $\tilde{\rho}^{\mathtt{S}}_{\pi,l}$ to have the same underlying $\overline{\mathbb{Q}}_l$-vector space as

$$\rho^{\mathtt{S}}_{\pi,l} := \bigotimes_{\mathfrak{p}\in\Sigma_p}\Big(\bigotimes_{\Sigma_{\infty/\mathfrak{p}} - \mathtt{S}_{\infty/\mathfrak{p}}} \text{-}\mathrm{Ind}^{\mathrm{Gal}_{\mathbb{Q}_p}}_{\mathrm{Gal}_{F_\mathfrak{p}}} \rho_{\pi,l}|_{\mathrm{Gal}_{F_\mathfrak{p}}}\Big),$$

on which $\Phi_{\mathfrak{p}^2}^{n_\mathfrak{p}}$ acts as $p^{2n_\mathfrak{p}}$-th (geometric) Frobenius $\mathrm{Frob}_{p^{2n_\mathfrak{p}}}$ times the number $\omega_\pi(\varpi_\mathfrak{p})^{n_\mathfrak{p}(1-\#\mathtt{S}_{\infty/\mathfrak{p}}/d_\mathfrak{p})}$ on the factor $\bigotimes -\mathrm{Ind}^{\mathrm{Gal}_{\mathbb{Q}_p}}_{\mathrm{Gal}_{F_\mathfrak{p}}}(\rho_{\pi,l}|_{\mathrm{Gal}_{F_\mathfrak{p}}})$, and acts trivially on the other factors. Here, $\varpi_\mathfrak{q} \in \mathbb{A}_E^\times$ is an idèle which is a uniformizer at $\mathfrak{q}$ and 1 otherwise. The action of $(\Phi_{\mathfrak{p}^2})^{n_\mathfrak{p}}$ on the $\overline{\mathbb{Q}}_l(-1)$'s indexed by $\Sigma_{\infty/\mathfrak{p}} - \mathtt{S}_{\infty/\mathfrak{p}}$ is the multiplication by p^2, and is trivial on the other $\overline{\mathbb{Q}}_l(-1)$'s.

Remark 5.19. — This Conjecture provides certain refinement of Langlands' philosophy on describing Galois representation appearing in the cohomology of Shimura varieties. Unfortunately, to our best knowledge of literature, only the action of "total Frobenius" was addressed using trace formula. It might be possible to modify the proof to understand the action of partial Frobenius; but this would go beyond the scope this paper. We leave it as a conjecture for interesting readers to pursue.

An alternative way to establish such a result is to generalize the Eichler-Shimura relations to our case. We are informed that Cornut and Nekovář have made progress along this line, but we do not know their hypothesis on p.

Conjecture 5.18 holds when we take the product of all twisted partial Frobenii.

Proposition 5.20. — *Put $d_\wp = [k_\wp : \mathbb{F}_p]$ and let $\Phi_{\wp^2}$ denote the product $\prod_{\mathfrak{p}\in\Sigma_p} \Phi_{\mathfrak{p}^2}^{d_\wp}$. We fix a Hecke character $\chi \in \mathscr{A}^w_{E,\tilde{\Sigma}}$.*

1. *Then the equality* (5.18.1) *holds in the Grothendieck group of modules of* $\overline{\mathbb{Q}}_l[\mathrm{GL}_2(\mathbb{A}^{\infty,p})][\Phi_{\wp^2}]$. *Here, for each* π *in* $\mathscr{A}_{(\underline{k},w)}[\chi_F]$ *or in* $\mathscr{B}_w[\chi_F]$, $\Phi_{\wp^2}$ *acts on* $\tilde{\rho}^{\mathtt{S}}_{\pi,l}$ *as* $\rho^{\mathtt{S}}_{\pi,l}(\mathrm{Frob}^2_{\wp})$ *multiplied by*
$$\omega_\pi(\underline{p})^{d_\wp}\cdot\prod_{\mathfrak{q}\in\Sigma_{E,p}}\chi(\varpi_{\mathfrak{q}})^{-2d_\wp\#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}/d_{\mathfrak{q}}}$$
where $\underline{p}\in\mathbb{A}_F^\times$ *denotes the idèle element* u *which equals* p *at all* p*-adic places and* 1 *otherwise.*
2. *Assume that for any* p*-adic place* $\mathfrak{p}\in\Sigma_p$ *that splits into two primes* $\mathfrak{q}$ *and* $\bar{\mathfrak{q}}$ *in* E, *we have* $\#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}=\#\tilde{\mathtt{S}}_{\infty/\bar{\mathfrak{q}}}$. *Then the number in* (1) *is equal to*
$$\omega_\pi(\underline{p})^{d_\wp}\cdot\prod_{\mathfrak{p}\in\Sigma_p}\omega_\pi(\varpi_{\mathfrak{p}})^{-d_\wp\#\mathtt{S}_{\infty/\mathfrak{p}}/d_{\mathfrak{p}}}=u\cdot p^{(w-2)(\#\mathtt{S}_\infty-g)d_\wp},$$
for u *some root of unity.*
3. *Conjecture 5.18 holds if* p *is inert in* F.

Proof. — (1) Combining (5.16.1) and Theorem 5.12, we get an equality in the Grothendieck group of modules over $\overline{\mathbb{Q}}_l[\mathrm{GL}_2(\mathbb{A}^{\infty,p})][\Phi_{\wp^2}]$:
$$\begin{aligned}&\big[H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{F}}_p},\mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma},l})^{\mathbb{A}_E^{\infty,\times}=\chi}\big]\\ =&\big[H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K_{\mathtt{S},p}}(G_{\mathtt{S}})_{\overline{\mathbb{F}}_p},\mathscr{L}^{(\underline{k},w)}_{\mathtt{S},l})^{\mathbb{A}_F^{\infty,\times}=\chi_F}\otimes\rho_{\chi,\tilde{\mathtt{S}},l}\big]\\ =&(-1)^{g-\#\mathtt{S}_\infty}\sum_{\pi\in\mathscr{A}_{(\underline{k},w)}[\chi_F]}\big[(\pi^\infty_{\mathtt{S}})^{K_{\mathtt{S},p}}\otimes\rho^{\mathtt{S}}_{\pi,l}\otimes\rho_{\chi,\tilde{\mathtt{S}},l}\big]\\ &+\delta_{\underline{k},2}\sum_{\pi\in\mathscr{B}_w[\chi_F]}\big[(\pi^\infty_{\mathtt{S}})^{K_{\mathtt{S},p}}\otimes\rho^{\mathtt{S}}_{\pi,l}\otimes\rho_{\chi,\tilde{\mathtt{S}},l}\big]\otimes\Big(\big[(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l(-1))^{\otimes(\Sigma_\infty-\mathtt{S}_\infty)}\big]-\delta_{\mathtt{S},\emptyset}\big[\overline{\mathbb{Q}}_l\big]\Big)\end{aligned}$$
Note that $\Phi_{\wp^2}$ acts on the cohomology as $\mathrm{Frob}^2_\wp\cdot S_p^{-d_\wp}$ by Subsection 5.8(4). Let $\mathrm{Frob}_{\tilde{\wp}}$ denote the geometric Frobenius element of the residue field $k_{\tilde{\wp}}$ of $\mathcal{O}_{\tilde{\wp}}$. We have either $\mathrm{Frob}_{\tilde{\wp}}=\mathrm{Frob}_\wp$ and $\mathrm{Frob}_{\tilde{\wp}}=\mathrm{Frob}^2_\wp$. In both cases, it follows from Lemma 5.15
$$\rho_{\chi,\tilde{\mathtt{S}},l}(\mathrm{Frob}^2_\wp)=\prod_{\mathfrak{q}\in\Sigma_{E,p}}\chi(\varpi_{\mathfrak{q}})^{-2d_\wp\#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}/d_{\mathfrak{q}}}.$$
Since the action of S_p is the given by the central idèle element $\underline{p}^{-1}\in\mathbb{A}_F^\times$, $\Phi_{\wp^2}$ acts on $(\pi^\infty_{\mathtt{S}})^{K_{\mathtt{S},p}}\otimes\rho^{\mathtt{S}}_{\pi,l}\otimes\rho_{\chi,\tilde{\mathtt{S}},l}$ for each $\pi\in\mathscr{A}_{(\underline{k},w)}[\chi_F]$ as $\rho^{\mathtt{S}}_{\pi,l}(\mathrm{Frob}^2_\wp)$ multiplied by
$$\omega_\pi(\underline{p}^{d_\wp})\prod_{\mathfrak{q}\in\Sigma_{E,p}}\chi(\varpi_{\mathfrak{q}})^{-2d_\wp\#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}/d_{\mathfrak{q}}}$$
Similarly, one proves the statement for $\pi\in\mathscr{B}_w[\chi_F]$.

(2) Let $\mathfrak{p}\in\Sigma_p$. If there is a unique prime $\mathfrak{q}$ of E above $\mathfrak{p}$, we have $d_{\mathfrak{q}}=2d_{\mathfrak{p}}$ and $\chi(\varpi_{\mathfrak{q}})=\omega_\pi(\varpi_{\mathfrak{p}})$ since $\omega_\pi=\chi_F$. If $\mathfrak{p}$ splits into $\mathfrak{q}$ and $\bar{\mathfrak{q}}$, under the assumption of the statement, we have $\#\mathtt{S}_{\infty/\mathfrak{q}}=\#\mathtt{S}_{\infty/\bar{\mathfrak{q}}}=\frac{1}{2}\#\mathtt{S}_{\infty/\mathfrak{p}}$. It follows immediately that the number to be multiplied in (1) is
$$\omega_\pi(\underline{p}^{d_\wp})\prod_{\mathfrak{p}\in\Sigma_p}\omega_\pi(\varpi_{\mathfrak{p}})^{-d_\wp\#\mathtt{S}_{\infty/\mathfrak{p}}/d_{\mathfrak{p}}}=\Big(\varepsilon_\pi(\underline{p})^{d_\wp}\prod_{\mathfrak{p}\in\Sigma_p}\varepsilon_\pi(\varpi_{\mathfrak{p}})^{-d_\wp\#\mathtt{S}_{\infty/\mathfrak{p}}/d_{\mathfrak{p}}}\Big)p^{d_\wp(w-2)(\#\mathtt{S}_\infty-g)}.$$
Here, $\varepsilon_\pi:=\omega_\pi|\cdot|_F^{2-w}$ is a Hecke character of finite order, hence the expression in the bracket is a root of unity.

(3) The last statement is clear because, when p is inert, Conjecture 5.18 is the same statement as what we have just proved. □

5.21. Description of GO-stratification of $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{\mathbb{F}_p}$. — Let k_0 be a finite field containing all residue fields of $\mathcal{O}_E$ of characteristic p. The main result of [**TX13a**] says that the GO-stratum $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0,\mathtt{T}}$, for $\mathtt{T}\subseteq\Sigma_\infty$, is naturally isomorphic to a $\mathbb{P}^1$-power bundle over the special fiber of another Shimura variety: $\mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{k_0}$, for some appropriate $\tilde{\mathtt{S}}(\mathtt{T})$. We now recall this result in more details as follows.

We recall first the definition of $\mathtt{S}(\mathtt{T})\subseteq\Sigma_\infty\cup\Sigma_p$ given in [**TX13a**, 5.1] for our case (i.e., $\mathtt{S}=\emptyset$ using the notation from *loc. cit.*) It suffices to specify $\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}=\mathtt{S}(\mathtt{T})\cap(\Sigma_{\infty/\mathfrak{p}}\cup\{\mathfrak{p}\})$ for each $\mathfrak{p}\in\Sigma_p$, since $\mathtt{S}(\mathtt{T})=\bigcup_{\mathfrak{p}\in\Sigma_p}\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}$. According to the convention of *loc. cit.*, we have several cases:

(Case $\alpha1$) $[F_\mathfrak{p}:\mathbb{Q}_p]$ is even, and $\mathtt{T}_\mathfrak{p}:=\mathtt{T}\cap\Sigma_{\infty/\mathfrak{p}}\subsetneq\Sigma_{\infty/\mathfrak{p}}$. In this case, we write $\mathtt{T}_\mathfrak{p}=\coprod C_i$ as disjoint union of chains. Here, by a chain, we mean there exists a $\tau_i\in\Sigma_{\infty/\mathfrak{p}}$ and an integer $r_i\geq 1$ such that $C_i=\{\sigma^{-a}\tau_i : 1\leq a\leq r_i\}$ with $\tau_i,\sigma^{-r_i+1}\tau_i\notin\mathtt{S}_{\infty/\mathfrak{p}}$. For each cycle C_i, if r_i is even, we put $C'_i=C_i$; if r_i is odd, we put $C'_i=C_i\cup\{\sigma^{-r_i}\tau_i\}$. Then, we define $\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}=\coprod C'_i$.

For example, if $\Sigma_{\infty/\mathfrak{p}}=\{\tau_0,\sigma\tau_0,\dots,\sigma^5\tau_0\}$ and $\mathtt{S}_{/\mathfrak{p}}=\{\sigma\tau_0,\sigma^3\tau_0,\sigma^4\tau_0\}$, then we have $\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}=\{\tau_0,\sigma\tau_0,\sigma^3\tau_0,\sigma^4\tau_0\}$.

(Case $\alpha2$) $[F_\mathfrak{p}:\mathbb{Q}_p]$ is even and $\mathtt{T}_\mathfrak{p}=\Sigma_{\infty/\mathfrak{p}}$. In this case, we put $\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}=\Sigma_{\infty/\mathfrak{p}}$.

(Case $\beta1$) $[F_\mathfrak{p}:\mathbb{Q}_p]$ is odd and $\mathtt{T}_\mathfrak{p}\subsetneq\Sigma_{\infty/\mathfrak{p}}$. In this case, we define $\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}$ in the same rule as Case $\alpha1$.

(Case $\beta2$) $[F_\mathfrak{p}:\mathbb{Q}_p]$ is odd and $\mathtt{T}_\mathfrak{p}=\Sigma_{\infty/\mathfrak{p}}$. We put $\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}=\Sigma_{\infty/\mathfrak{p}}\cup\{\mathfrak{p}\}$.

It is clear from the definition that $\sigma_\mathfrak{p}(\mathtt{S}(\mathtt{T}))=\mathtt{S}(\sigma_\mathfrak{p}(\mathtt{T}))$.

We do not recall the precise choice of the lifts $\tilde{\mathtt{S}}(\mathtt{T})_\infty$, as it is combinatorially complicated. We refer interesting readers to loc. cit. *for the construction. In this paper, we only need to know that $\tilde{\mathtt{S}}(\mathtt{T})_\infty$ satisfies the condition in Proposition 5.20(2), i.e. if a prime $\mathfrak{p}\in\Sigma_p$ splits into two places $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ in E, then $\#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}=\#\tilde{\mathtt{S}}_{\infty/\bar{\mathfrak{q}}}$.*

To specify the subgroup $K''_{\mathtt{S}(\mathtt{T}),p}\subset G''_{\tilde{\mathtt{S}}(\mathtt{T})}(\mathbb{Q}_p)$, we define first a subgroup $K_{\mathtt{S}(\mathtt{T}),p}=\prod_{\mathfrak{p}\in\Sigma_p}K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}\subset G_{\mathtt{S}(\mathtt{T})}(\mathbb{Q}_p)=\prod_{\mathfrak{p}\in\Sigma_p}(B_\mathtt{S}\otimes_F F_\mathfrak{p})^\times$ as follows. We fix an isomorphism $(B_{\mathtt{S}(\mathtt{T})}\otimes_F F_\mathfrak{p})^\times\simeq\mathrm{GL}_2(F_\mathfrak{p})$ for each $\mathfrak{p}\notin\mathtt{S}(\mathtt{T})$, i.e. if we are in cases $\alpha1$, $\alpha2$ and $\beta1$.

- In Case $\alpha1$ and $\beta1$, we take $K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}=\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$.
- In Case $\alpha2$, we take $K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}=\mathrm{Iw}_\mathfrak{p}$, where $\mathrm{Iw}_\mathfrak{p}\subset\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ is the Iwahoric subgroup (3.3.1).
- In Case $\beta2$, $B_{\mathtt{S}(\mathtt{T})}$ is ramified at $\mathfrak{p}$, and we take $K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}=\mathcal{O}^\times_{B_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}$. Here, $\mathcal{O}_{B_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}$ denotes the unique maximal order of $B_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}=B_{\mathtt{S}(\mathtt{T})}\otimes_F F_\mathfrak{p}$.

We put then
$$K''_{\mathtt{S}(\mathtt{T}),p}=K_{\mathtt{S}(\mathtt{T}),p}\times_{(\mathcal{O}_F\otimes\mathbb{Z}_p)^\times}(\mathcal{O}_E\otimes\mathbb{Z}_p)^\times\subset G''_{\mathtt{S}(\mathtt{T})}(\mathbb{Q}_p).$$

Now we can recall the main result of [**TX13a**] in the case for $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_\emptyset)_{k_0}$.

***Theorem 5.22* (*loc. cit.* Corollary 5.11).** — 1. *The GO-stratum $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_\emptyset)_{k_0,\mathtt{T}}$ is isomorphic to a $(\mathbb{P}^1)^{I_\mathtt{T}}$-bundle over $\mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{k_0}$, where the index set is*
$$I_\mathtt{T}=\mathtt{S}(\mathtt{T})_\infty-\mathtt{T}=\bigcup_{\mathfrak{q}\in\Sigma_p}\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{q}}-\mathtt{T}_\mathfrak{q}.$$

2. *The natural projection $\pi_\mathtt{T}:\mathbf{Sh}_{K''_{\emptyset,p}}(G''_\emptyset)_{k_0,\mathtt{T}}\to\mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{k_0}$ given in* (1) *is equivariant for the action of $G''_\emptyset(\mathbb{A}^{\infty,p})\cong G''_{\tilde{\mathtt{S}}(\mathtt{T})}(\mathbb{A}^{\infty,p})$. In particular, for a sufficiently small compact subgroup $K''^p_\emptyset\subset G''_\emptyset(\mathbb{A}^{\infty,p})$, the projection $\pi_\mathtt{T}$ descends to a $(\mathbb{P}^1)^{I_\mathtt{T}}$-fibration $\pi_\mathtt{T}\colon\mathbf{Sh}_{K''_\emptyset}(G''_\emptyset)_{k_0,\mathtt{T}}\to\mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T})}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{k_0}$, where $K''_\emptyset=K''^p_\emptyset K''_{\emptyset,p}$ and $K''_{\mathtt{S}(\mathtt{T})}=K''^p_\emptyset K''_{\mathtt{S}(\mathtt{T}),p}$. Here, we have fixed an isomorphism between $G''_\emptyset(\mathbb{A}^{\infty,p})$ and $G''_{\mathtt{S}(\mathtt{T})}(\mathbb{A}^{\infty,p})$, and view also $K''^p_\emptyset$ also as a subgroup of the latter.*

3. *Let* $\mathbf{A}''_{\emptyset,k_0}$ *(resp.* $\mathbf{A}''_{\tilde{\mathtt{S}}(\mathtt{T}),k_0}$*) be the family of abelian varieties over* $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0}$ *(resp.* $\mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{k_0}$*) discussed in Subsection 5.8. Then the restriction of* $\mathbf{A}''_{\emptyset,k_0}$ *to* $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0,\mathtt{T}}$ *is isogenous to* $\pi_{\mathtt{T}}^*(\mathbf{A}''_{\tilde{\mathtt{S}}(\mathtt{T}),k_0})$.
4. *For each* $\mathfrak{p}\in\Sigma_p$*, we have a commutative diagram:*

$$\begin{array}{ccccc}
 & & \xrightarrow{\qquad\qquad\mathfrak{F}_{\mathfrak{p}^2,\emptyset}\qquad\qquad} & & \\
\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0,\mathtt{T}} & \xrightarrow{\xi^{\mathrm{rel}}} & \mathfrak{F}^*_{\mathfrak{p}^2,\mathtt{S}(\mathtt{T})}(\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0,\sigma_{\mathfrak{p}}^2\mathtt{T}}) & \xrightarrow{\mathfrak{F}^*_{\mathfrak{p}^2,\mathtt{S}(\mathtt{T})}} & \tilde{\mathbf{Sh}}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0,\sigma_{\mathfrak{p}}^2\mathtt{T}} \\
 & \searrow^{\pi_{\mathtt{T}}} & \downarrow & & \downarrow{\scriptstyle \pi_{\sigma_{\mathfrak{p}}^2\mathtt{T}}} \\
 & & \mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{k_0} & \xrightarrow{\mathfrak{F}_{\mathfrak{p}^2,\tilde{\mathtt{S}}(\mathtt{T})}} & \mathbf{Sh}_{K''_{\sigma_{\mathfrak{p}}^2\mathtt{S}(\mathtt{T}),p}}(G''_{\sigma_{\mathfrak{p}}^2\tilde{\mathtt{S}}(\mathtt{T})})_{k_0}
\end{array}$$

where the square is cartesian, $\mathfrak{F}''_{\mathfrak{p}^2,\emptyset}$ *(resp.* $\mathfrak{F}''_{\mathfrak{p}^2,\mathtt{S}(\mathtt{T})}$*) is the twisted partial Frobenius on* $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0}$ *(resp.* $\mathbf{Sh}_{K''_{\tilde{\mathtt{S}}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{k_0}$*)* [**TX13a**, 3.23], *and* ξ^{rel} *is a morphism whose restriction to a fiber* $\pi_{\mathtt{T}}^{-1}(x)=(\mathbb{P}^1_x)^{I_{\mathtt{T}}}$ *is the product of the relative* p^2*-Frobenius of the* $\mathbb{P}^1_x$*'s indexed by* $I_{\mathtt{T}}\cap\Sigma_{\infty/\mathfrak{p}}=\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}-\mathtt{T}_{\mathfrak{p}}$*, and the identity on the other* $\mathbb{P}^1_x$*'s.*

We list a few special cases of the theorem for the convenience of the readers.

Example 5.23. — The prime-to-p level of all Shimura varieties below are taken to be the same as $K''^{p}_{\emptyset}$ (they can be naturally identified); unless specified otherwise, the level structure K''_p at p is taken to be "maximal". To simplify notation, we use $\tilde{X}_{\mathtt{T}}$ to denote the GO-stratum $\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{k_0,\mathtt{T}}$ and $\overline{Sh}_{\mathtt{S}}$ to denote the Shimura variety $\mathbf{Sh}_{K''_{\mathtt{S},p}}(G''_{\tilde{\mathtt{S}}})_{k_0}$ (note that we have suppressed the choice of signature here.)

(1) When F is a real quadratic field in which p splits into two places $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the chosen isomorphism $\iota_p:\mathbb{C}\xrightarrow{\cong}\overline{\mathbb{Q}}_p$ associates to each place $\mathfrak{p}_i$ an archimedean place ∞_i of F. Then the non-trivial closed GO-strata are $\tilde{X}_{\{\infty_1\}}$, $\tilde{X}_{\{\infty_2\}}$, and $\tilde{X}_{\{\infty_1,\infty_2\}}$. Then Theorem 5.22 says that each $\tilde{X}_{\{\infty_i\}}$ is isomorphic to $\overline{Sh}_{\{\mathfrak{p}_i,\infty_i\}}$, and $\tilde{X}_{\{\infty_1,\infty_2\}}$ is isomorphic to $\overline{Sh}_{\{\mathfrak{p}_1,\mathfrak{p}_2,\infty_1,\infty_2\}}$.
(2) When F is a real quadratic field in which p is inert, we label the two archimedean places of F to be ∞_1 and ∞_2. Then Theorem 5.22 says that each $\tilde{X}_{\{\infty_i\}}$ is isomorphic to a $\mathbb{P}^1$-bundle over $\overline{Sh}_{\{\infty_1,\infty_2\}}$, and $\tilde{X}_{\{\infty_1,\infty_2\}}$ is isomorphic to $\overline{Sh}_{\{\infty_1,\infty_2\}}$ with an *Iwahori level structure at* p.
(3) When F is a real cubic field in which p is inert, the chosen isomorphism $\iota_p:\mathbb{C}\xrightarrow{\cong}\overline{\mathbb{Q}}_p$ makes $\Sigma_\infty=\{\infty_0,\infty_1,\infty_2\}$ into a cycle under the action of the Frobenius σ, i.e. $\infty_0\overset{\sigma}{\mapsto}\infty_1\overset{\sigma}{\mapsto}\infty_2\overset{\sigma}{\mapsto}\infty_3=\infty_0$. The stratum $\tilde{X}_{\{\infty_i\}}$ is isomorphic to a $\mathbb{P}^1$-bundle over $\overline{Sh}_{\{\infty_{i-1},\infty_i\}}$; the stratum $\tilde{X}_{\{\infty_{i-1},\infty_i\}}$ is isomorphic to $\overline{Sh}_{\{\infty_{i-1},\infty_i\}}$; the stratum $\tilde{X}_{\{\infty_1,\infty_2,\infty_3\}}$ is isomorphic to $\overline{Sh}_{\{p,\infty_1,\infty_2,\infty_3\}}$.
(4) When F is a totally real field of degree 4 over $\mathbb{Q}$ in which p is inert, we may label the archimedean places of F to be $\infty_1,\dots,\infty_4$ such that the Frobenius σ takes each ∞_i to ∞_{i+1}, where $\infty_i=\infty_{i\bmod 4}$. We have the following list of the non-trivial strata.

Strata	Description
$\tilde{X}_{\{\infty_i\}}$ for each i	$\mathbb{P}^1$-bundle over $\overline{Sh}_{\{\infty_{i-1},\infty_i\}}$
$\tilde{X}_{\{\infty_{i-1},\infty_i\}}$ for each i	$\overline{Sh}_{\{\infty_{i-1},\infty_i\}}$
$\tilde{X}_{\{\infty_1,\infty_3\}}$ and $\tilde{X}_{\{\infty_2,\infty_4\}}$	$(\mathbb{P}^1)^2$-bundle over $\overline{Sh}_{\{\infty_1,\dots,\infty_4\}}$
$\tilde{X}_{\mathtt{T}}$ with $\#\mathtt{T}=3$	$\mathbb{P}^1$-bundle over $\overline{Sh}_{\{\infty_1,\dots,\infty_4\}}$
$\tilde{X}_{\{\infty_1,\dots,\infty_4\}}$	$\overline{Sh}_{\{\infty_1,\dots,\infty_4\}}$ with Iwahori level at p

We fix an open compact $K = K^pK_p \subset \mathrm{GL}_2(\mathbb{A}^\infty)$ with $K_p = \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p)$ and K^p sufficiently small. For $\mathtt{T} \subseteq \Sigma_\infty$, we denote by $Y_{\mathtt{T}}$ the closed GO-stratum $\mathbf{Sh}_K(G)_{k_0,\mathtt{T}}$ as in Subsection 4.4. We put $K_{\mathtt{S}(\mathtt{T})} = K^pK_{\mathtt{S}(\mathtt{T}),p}$ with $K_{\mathtt{S}(\mathtt{T}),p}$ just defined before Theorem 5.22. Here, we fix an isomorphism $G_{\mathtt{S}(\mathtt{T})}(\mathbb{A}^{\infty,p}) \simeq \mathrm{GL}_2(\mathbb{A}^{\infty,p})$, and regard K^p as an subgroup of $G_{\mathtt{S}(\mathtt{T})}(\mathbb{A}^{\infty,p})$. We will put all the results in this section together to computer the cohomology of $Y_{\mathtt{T}}$.

Proposition 5.24. — *Let $F_{\mathtt{S}(\mathtt{T})}$ be the reflex field of $\mathbf{Sh}_{K_{\mathtt{S}(\mathtt{T})}}(G_{\mathtt{S}(\mathtt{T})})$, $k_\wp$ be the residue field of $\mathcal{O}_{F_{\mathtt{S}(\mathtt{T})}}$ at the p-adic place given by the isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$, and $d_\wp = [k_\wp : \mathbb{F}_p]$. In the Grothendieck group of modules of $\mathscr{H}(K^p, \overline{\mathbb{Q}}_l)[S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1} : \mathfrak{p} \in \Sigma_p][\Phi_{\wp^2}]$, we have an equality*

(5.24.1)

$$[H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p}, \mathscr{L}_l^{(\underline{k},w)})] = (-1)^{g-\#\mathtt{S}(\mathtt{T})_\infty}\Big[\bigoplus_{\pi \in \mathscr{A}_{(\underline{k},w)}} (\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}} \otimes \tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l} \otimes (\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l(-1))^{\otimes I_{\mathtt{T}}}\Big]$$
$$+ \delta_{\underline{k},2}\Big[\bigoplus_{\pi \in \mathscr{B}_w} (\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}} \otimes \tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}\Big] \otimes \Big(\big[(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l(-1))^{\otimes(\Sigma_\infty - \mathtt{T})}\big] - \delta_{\mathtt{T},\emptyset}[\overline{\mathbb{Q}}_l]\Big),$$

where

- *$\delta_{\underline{k},2}$ is equal to 1 if all $k_\tau = 2$, and 0 otherwise,*
- *$\delta_{\mathtt{T},\emptyset}$ is equal to 1 if $\mathtt{T} = \emptyset$, and 0 otherwise,*
- *On each $\overline{\mathbb{Q}}_l(-1)$, $\mathscr{H}(K^p, \overline{\mathbb{Q}}_l)[S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1}; \mathfrak{p} \in \Sigma_p]$ acts trivially, and $\Phi_{\wp^2}$ acts by multiplication by $p^{2d_\wp}$.*
- *$\Phi_{\wp^2}$ acts on the left hand side of (5.24.1) by $\prod_{\mathfrak{p} \in \Sigma_p} (\Phi_{\mathfrak{p}^2})^{d_\wp}$ with $\Phi_{\mathfrak{p}^2} = \mathrm{Fr}_{\mathfrak{p}}^2 \cdot S_{\mathfrak{p}}^{-1}$ as defined in (4.9.1).*
- *for $\pi \in \mathscr{A}_{(\underline{k},w)}$ or $\pi \in \mathscr{B}_w$, $\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}$ is isomorphic to $\rho^{\mathtt{S}(\mathtt{T})}_{\pi,l}$ as a vector space, and is equipped with a $\Phi_{\wp^2}$-action given by $\rho^{\mathtt{S}(\mathtt{T})}_{\pi,l}(\mathrm{Frob}_\wp)^2$ multiplied by the number*
$$\omega_\pi(\underline{p})^{d_\wp} \cdot \prod_{\mathfrak{p} \in \Sigma_p} \omega_\pi(\varpi_{\mathfrak{p}})^{-d_\wp \#\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}/d_{\mathfrak{p}}} = u \cdot p^{(w-2)(\#\mathtt{S}(\mathtt{T})_\infty - g)d_\wp},$$
with u a root of unity. Note that when $\pi \in \mathscr{B}_w$, $\Phi_{\wp^2}$ acts trivially on $\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}$.

Moreover, if Conjecture 5.18 holds, the equality (5.24.1) *holds in the Grothendieck group of modules of $\mathscr{H}(K^p, \overline{\mathbb{Q}}_l)[\Phi_{\mathfrak{p}^2}^{n_{\mathfrak{p}}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1}; \mathfrak{p} \in \Sigma_p]$, where*

- *$n_{\mathfrak{p}}$ is the smallest positive number n such that $\sigma_{\mathfrak{p}}^n\mathtt{T} = \mathtt{T}$,*
- *In $(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l(-1))^{\otimes I_{\mathtt{T}}}$ and $(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l(-1))^{\otimes(\Sigma_\infty - \mathtt{T})}$, $\Phi_{\mathfrak{p}^2}$ acts trivially on $\overline{\mathbb{Q}}_l$ and on the copies of $\overline{\mathbb{Q}}_l(-1)$ which are labeled with elements not in $\Sigma_{\infty/\mathfrak{p}}$; on the copies of $\overline{\mathbb{Q}}_l(-1)$'s labeled by elements in $\Sigma_{\infty/\mathfrak{p}}$, the action of $\Phi_{\mathfrak{p}^2}^{n_{\mathfrak{p}}}$ is the multiplication by $p^{2n_{\mathfrak{p}}}$.*
- *$\Phi_{\mathfrak{p}^2}^{n_{\mathfrak{p}}}$ acts on $\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}$ (resp. $\tilde{\rho}_{\pi,l}$) by $p^{2n_{\mathfrak{p}}}$-Frobenius at $\mathfrak{p}$ on $\rho^{\mathtt{S}(\mathtt{T})}_{\pi,l}$ (resp. $\rho_{\pi,l}$) times the number $\omega_\pi(\varpi_{\mathfrak{p}})^{-n_{\mathfrak{p}}(1-\#\mathtt{S}_{\infty/\mathfrak{p}}/d_{\mathfrak{p}})}$.*

Proof. — We first remark that the Hecke action of $F^\times$ (viewed as a subgroup of the center $\mathbb{A}_F^{\infty,\times} \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$) on $H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p}, \mathscr{L}_l^{(\underline{k},w)})$ is given by $(2-w)$nd power of the norm. Hence, to prove the equality above, we may consider the submodules on both sides on which $\mathbb{A}_F^{\infty,\times}$ acts via the restriction of a fixed Hecke character χ_F of F whose all archimedean components are given by $x \mapsto x^{w-2}$. By the discussion in Notation 5.17, there exists a Hecke character $\chi \in \mathscr{A}^w_{E,\tilde{\Sigma}}$ whose restriction is just χ_F. Then it follows from (5.16.2) that

$$H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p}, \mathscr{L}_l^{(\underline{k},w)})^{\mathbb{A}_F^{\infty,\times}=\chi_F} = H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K_p}(G)_{\overline{\mathbb{F}}_p,\mathtt{T}}, \mathscr{L}_l^{\underline{k},w})^{\mathbb{A}_F^{\infty,\times}=\chi_F, K^p=\mathrm{trivial}} \tag{5.24.2}$$
$$= H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_{\emptyset,p}}(G''_\emptyset)_{\overline{\mathbb{F}}_p,\mathtt{T}}, \mathscr{L}''^{(\underline{k},w)}_{\emptyset,\tilde{\Sigma},l})^{\mathbb{A}_E^{\infty,\times}=\chi, K^p=\mathrm{trivial}}.$$

By Theorem 5.22(1) and (2), we see that $\mathscr{L}''^{(\underline{k},w)}_{\emptyset,\tilde{\Sigma},l}|_{\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{\mathbb{F}_p,\mathtt{T}}} \simeq \pi_{\mathtt{T}}^*(\mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}}(\mathtt{T}),\tilde{\Sigma},l})$, and hence

(5.24.3)
$$R^n\pi_{\mathtt{T},*}\big(\mathscr{L}''^{(\underline{k},w)}_{\emptyset,\tilde{\Sigma},l}|_{\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{\mathbb{F}_p,\mathtt{T}}}\big) = \begin{cases} \mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}}(\mathtt{T}),\tilde{\Sigma},l} \otimes \overline{\mathbb{Q}}_l(-i)^{\oplus\binom{\#I_{\mathtt{T}}}{i}} & \text{if } n=2i \text{ with } 0\leq i\leq \#I_{\mathtt{T}};\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, in the Grothendieck group of $\mathscr{H}(K^p,\overline{\mathbb{Q}}_l)[S_{\mathfrak{p}},S_{\mathfrak{p}}^{-1}:\mathfrak{p}\in\Sigma_p][\Phi_{\wp^2}]$-modules, we have

$$\begin{aligned}
&[H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}^{(\underline{k},w)}_l)^{\mathbb{A}_F^{\infty,\times}=\chi_F}]\\
&=\big[H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{\overline{\mathbb{F}}_p},R\pi_{\mathtt{T},*}\big(\mathscr{L}''^{(\underline{k},w)}_{\emptyset,\tilde{\Sigma},l}|_{\mathbf{Sh}_{K''_{\emptyset,p}}(G''_{\emptyset})_{\mathbb{F}_p,\mathtt{T}}}\big))^{\mathbb{A}_E^{\infty,\times}=\chi,K^p=\mathrm{trivial}}\big]\\
&=\big[H^\star_{c,\mathrm{et}}(\mathbf{Sh}_{K''_{\mathtt{S}(\mathtt{T}),p}}(G''_{\tilde{\mathtt{S}}(\mathtt{T})})_{\overline{\mathbb{F}}_p},\mathscr{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}}(\mathtt{T}),\tilde{\Sigma},l})^{\mathbb{A}_E^{\infty,\times}=\chi,K^p=\mathrm{trivial}}\big]\otimes\big[(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l(-1))^{\otimes I_{\mathtt{T}}}\big]\\
&=(-1)^{g-\#\mathtt{S}(\mathtt{T})_\infty}\Big[\bigoplus_{\pi\in\mathscr{A}_{(\underline{k},w)}[\chi_F]}(\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}}\otimes\tilde{\rho}^{\mathtt{S}}_{\pi,l}\otimes(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l(-1))^{\otimes I_{\mathtt{T}}}\Big]\\
&\quad+\delta_{\underline{k},2}\Big[\bigoplus_{\pi\in\mathscr{B}_w[\chi_F]}(\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}}\otimes\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}\Big]\otimes\Big(\big[(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l(-1))^{\otimes(\Sigma_\infty-\mathtt{S}(\mathtt{T})_\infty)}\big]-\delta_{\mathtt{S}(\mathtt{T}),\emptyset}\big[\overline{\mathbb{Q}}_l\big]\Big)\otimes\big[(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l(-1))^{\otimes I_{\mathtt{T}}}\big]
\end{aligned}$$

Here, we used Theorem 5.12 in the last equality. We remark that $\mathtt{S}(\mathtt{T})=\emptyset$ if and only if $\mathtt{T}=\emptyset$, and in which case $I_{\mathtt{T}}=\emptyset$. Now it is clear that the expression above is exactly the χ_F-component of (5.24.1). The description of the action of $\Phi_{\wp^2}$ is immediate from the fact that $\Phi_{\wp^2}=\mathrm{Frob}^2_{\wp}S_p^{-d_\wp}$ and Proposition 5.20.

The second part of the Theorem follows from exactly the same argument by using Conjecture 5.18 in place of Proposition 5.20. □

6. Computation of the Rigid Cohomology I

We will use the same notation as in Subsection 4.4 and Proposition 5.24. We consider the cohomology group $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})$ as defined in Subsection 3.4. Note that we have fixed an open compact subgroup $K=K^pK_p\subseteq\mathrm{GL}_2(\mathbb{A}_F^\infty)$ with $K_p=\mathrm{GL}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$, and omitted K from the notation. In this section, we will first use the results in Section 4 and 5 to compute it as an element in a certain Grothendieck group. Then, combining the results in Subsection 3, especially Proposition 3.25, we prove our main theorem on the classicality of overconvergent cusp forms.

The second part of the following theorem will not be used later, but it provide a good sense of what to expect and it is also a baby version of the computation in the next section. So we keep it here.

***Theorem 6.1* (Weak cohomology computation).** — *Let $(\underline{k},w)$ be a multiweight.*

1. *For each integer n, in the Grothendieck group of finitely generated modules over $\mathscr{H}(K^p,\overline{\mathbb{Q}}_p)$, $[H^n_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})\otimes_{L_\wp}\overline{\mathbb{Q}}_p]$ is a sum of Hecke modules coming from classical automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ (including cuspidal representations, one-dimensional representations and Eisenstein series) whose central character is an algebraic Hecke character with archimedean component $N^{w-2}_{F/\mathbb{Q}}$.*
2. *We have the following isomorphism of modules in the Grothendieck group of modules of $\mathscr{H}(K^p,\overline{\mathbb{Q}}_p)$.*

(6.1.1)
$$\big[H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})\otimes_{L_\wp}\overline{\mathbb{Q}}_p\big]=(-1)^g\cdot[S_{(\underline{k},w)}(K^p\mathrm{Iw}_p,\overline{\mathbb{Q}}_p)].$$

Proof. — (1) By the Hecke equivariant spectral sequence (4.10.1), each $H^n_{\mathrm{rig}}(X^{\mathrm{ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})$ is a sub-quotient of rigid cohomology groups of GO-strata $Y_{\mathtt{T}}$'s. It suffices to prove that, for all $\mathtt{T}\subseteq\Sigma_\infty$, each individual rigid cohomology group of $Y_{\mathtt{T}}$ is a sum of Hecke modules coming from classical automorphic representations. For $\mathtt{T}=\emptyset$, this is clear by standard comparison between rigid and de Rham cohomology and classical theory. For $\mathtt{T} \neq \emptyset$, we may reduce to a similar problem for étale cohomology of $Y_{\mathtt{T}}$ by Proposition 4.13. Then the required statement follows from Theorem 5.22 and the proof of Proposition 5.24. This proves statement (1).

(2) We also identify $\mathbb{C}$ with $\overline{\mathbb{Q}}_l$ via a fixed $\iota_l : \mathbb{C} \simeq \overline{\mathbb{Q}}_l$. Computing the tame Hecke action on the ordinary locus is straightforward:

$$\begin{aligned}
&\big[H^\star_{\mathrm{rig}}(Y^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)}) \otimes_{L_\wp} \overline{\mathbb{Q}}_p\big] \\
\xlongequal{(4.10.1)}\ & \sum_{\mathtt{T}\subseteq\Sigma_\infty} (-1)^{\#\mathtt{T}} \big[H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp, \mathscr{D}^{(\underline{k},w)}) \otimes_{L_\wp} \overline{\mathbb{Q}}_p\big] \\
\xlongequal{\text{Prop 4.13}}\ & \sum_{\mathtt{T}\subseteq\Sigma_\infty} (-1)^{\#\mathtt{T}} \big[H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p}, \mathscr{L}_l^{(\underline{k},w)}) \otimes_{L_\mathfrak{l}} \overline{\mathbb{Q}}_l\big] \\
\xlongequal{\text{Prop. 5.24}}\ & \sum_{\mathtt{T}\subseteq\Sigma_\infty} (-1)^{\#\mathtt{T}} \Big((-1)^{g-\#\mathtt{S}(\mathtt{T})_\infty} 2^{g-\#\mathtt{T}} \big[\bigoplus_{\pi\in\mathscr{A}_{(\underline{k},w)}} (\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}} \big] + \delta_{\underline{k},2} (2^{g-\#\mathtt{T}} - \delta_{\mathtt{T},\emptyset}) \big[\bigoplus_{\pi\in\mathscr{B}_w} (\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}} \big] \Big)
\end{aligned}$$

(6.1.2)

$$\begin{aligned}
= & \sum_{\pi\in\mathscr{A}_{(\underline{k},w)}} \big[(\pi^{\infty,p})^{K^p}\big] \prod_{\mathfrak{p}\in\Sigma_p} \sum_{\mathtt{T}_\mathfrak{p}\subseteq\Sigma_{\infty/\mathfrak{p}}} (-1)^{\#\mathtt{T}_\mathfrak{p}} (-1)^{\#\Sigma_{\infty/\mathfrak{p}} - \#\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}} \cdot 2^{\#\Sigma_{\infty/\mathfrak{p}} - \#\mathtt{T}_\mathfrak{p}} \big[(\pi_{\mathtt{S}(\mathtt{T}),\mathfrak{p}})^{K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}\big] \\
& + \delta_{\underline{k},2} \sum_{\pi\in\mathscr{B}_w} \big[(\pi^\infty)^K\big] \big(-1 + \sum_{\mathtt{T}\subseteq\Sigma_\infty} (-1)^{\#\mathtt{T}} 2^{g-\#\mathtt{T}} \big)
\end{aligned}$$

Here when quoting Proposition 5.24, we have turned the vector space for $\Phi_{\wp^2}$-action into their dimensions. The last equality is just to separate out the contribution from each prime $\mathfrak{p} \in \Sigma_p$ (note that $(\pi^{\infty,p}_{\mathtt{S}(\mathtt{T})})^{K^p_{\mathtt{S}(\mathtt{T})}}$ is isomorphic to $(\pi^{\infty,p})^{K^p}$).

We separate the computation for each $\pi \in \mathscr{A}_{(\underline{k},w)} \cup \mathscr{B}_w$.

For $\pi \in \mathscr{B}_w$, note that $\sum_{\mathtt{T}\subseteq\Sigma_\infty} (-1)^{\#\mathtt{T}} 2^{g-\#\mathtt{T}} = (2-1)^g = 1$ by binomial expansion. So the contribution of $\pi \in \mathscr{B}_w$ to $\big[H^\star_{\mathrm{rig}}(Y^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})\big]$ is trivial. This agrees with the right hand side of (6.1.1), where none of such π appears.

For $\pi \in \mathscr{A}_{(\underline{k},w)}$, we need to compute each factor of the product over $\mathfrak{p}$.

- When $\pi_\mathfrak{p}$ is not unramified, $(\pi_{\mathtt{S}(\mathtt{T}),\mathfrak{p}})^{K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}$ is nonzero if and only if $\mathtt{T}_\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}}$ and $\pi_\mathfrak{p}$ is Steinberg. In this case, $(\pi_{\mathtt{S}(\mathtt{T}),\mathfrak{p}})^{K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}$ is one dimensional. So the factor for $\mathfrak{p}$ in the product (6.1.2) is nontrivial only when $\mathtt{T}_\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}}$. It has total contribution of multiplicity $(-1)^{\#\Sigma_{\infty/\mathfrak{p}}}$ in this case.
- When $\pi_\mathfrak{p}$ is unramified, $(\pi_{\mathtt{S}(\mathtt{T}),\mathfrak{p}})^{K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}$ is one dimensional, unless $\mathtt{T}_\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}}$. In the latter case, it is zero if $\#\Sigma_{\infty/\mathfrak{p}}$ is odd and is 2 if $\Sigma_{\infty/\mathfrak{p}}$ is even, i.e. it is $1+(-1)^{\#\Sigma_{\infty/\mathfrak{p}}}$. Also note that $\#\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}$ is always even unless $\mathtt{S}(\mathtt{T})_{/\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}}$ and it is an odd set. But the latter case is exactly when $(\pi_{\mathtt{S}(\mathtt{T}),\mathfrak{p}})^{K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}$ vanishes. So we may ignore the term $(-1)^{\#\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}}$ in computation. The factor for $\mathfrak{p}$ in the product (6.1.2) has total contribution

$$\begin{aligned}
&\sum_{\mathtt{T}_\mathfrak{p}\subsetneq\Sigma_{\infty/\mathfrak{p}}} (-1)^{\#\mathtt{T}_\mathfrak{p}} (-1)^{\#\Sigma_{\infty/\mathfrak{p}}} \cdot 2^{\#\Sigma_{\infty/\mathfrak{p}} - \#\mathtt{T}_\mathfrak{p}} + \big(1 + (-1)^{\#\Sigma_{\infty/\mathfrak{p}}}\big) \\
=&\big((2-1)^{\#\Sigma_{\infty/\mathfrak{p}}} - 1\big) + \big(1 + (-1)^{\#\Sigma_{\infty/\mathfrak{p}}}\big) = 2 \times (-1)^{\#\Sigma_{\infty/\mathfrak{p}}}.
\end{aligned}$$

This means that the end contribution of $\pi_\mathfrak{p}$ from (4.10.1) agrees with its contribution to $S_{(\underline{k},w)}(K^p\mathrm{Iw}_p, \mathbb{C})$. Putting all places above p together proves the Theorem. □

Remark 6.2. — (1) By a careful check of cancelations in the spectral sequence (4.10.1), it is possible to show that each individual cohomology group $[H^n_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})]$ does not contain one-dimensional automorphic representations. However, it may indeed contain the tame Hecke spectrum of some Eisenstein series, because the Eisenstein spectrum in $H^n_{c,\mathrm{rig}}(X, \mathscr{D}^{(\underline{k},w)})$ can not be canceled out by cohomology groups of $Y_{\mathtt{T}}$ with $\mathtt{T} \neq \emptyset$.

(2) In the proof above, we have dropped the action of twisted partial Frobenius. We will get to a more delicate computation in the next subsection which involves matching the action of partial Frobenius with the action of $U_{\mathfrak{p}}$-operators.

We first arrive at the following very weak version of the classicality of overconvergent cusp forms, where we do not attempt to optimize the bound on slopes. The sole purpose is to prove that the classical cusp forms are Zariski dense in the Kisin-Lai eigencurve.

***Proposition 6.3* (Weak classicality).** — *Let $f \in S^{\dagger}_{(\underline{k},w)}(K,\overline{\mathbb{Q}}_p)$ be an overconvergent eigenform for $\overline{\mathbb{Q}}_p[U_{\mathfrak{p}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1} : \mathfrak{p}\in\Sigma_{\mathfrak{p}}]$. For $\mathfrak{p}\in\Sigma_p$, let $\lambda_{\mathfrak{p}}$ denote the eigenvalue of f for the operator $U_{\mathfrak{p}}$. Assume that*

$$\sum_{\mathfrak{p}\in\Sigma_p}\Big(\mathrm{val}_p(\lambda_{\mathfrak{p}}) - \sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}\frac{w-k_\tau}{2}\Big) + g < \frac{1}{g}\min_{\tau\in\Sigma_\infty}(k_\tau - 1). \tag{6.3.1}$$

Then f is a classical cusp form of level $K^p\mathrm{Iw}_p$.

Proof. — The basic idea is that, when the slope is very small comparing to the weights, there is essentially one term in the spectral sequence (4.10.1) which can possibly contribute to the corresponding slope. Moving between various normalizations unfortunately makes the proof look complicated.

We use superscript $\prod_{\mathfrak{p}} U_{\mathfrak{p}}$-slope $=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_p)$ to denote the subspace where the eigenvalues of $\prod_{\mathfrak{p}\in\Sigma_p}U_{\mathfrak{p}}$ all have p-adic valuation $\sum_{\mathfrak{p}\in\Sigma_p}\mathrm{val}_p(\lambda_p)$. We aim to show that, under the weight-slope condition (6.3.1), the natural embedding

$$S_{(\underline{k},w)}(K^p\mathrm{Iw}_p,\overline{\mathbb{Q}}_p)^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_p)} \hookrightarrow S^{\dagger}_{(\underline{k},w)}(K,\overline{\mathbb{Q}}_p)^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_p)} \tag{6.3.2}$$

is in fact an isomorphism. It suffices to show that both sides have the same dimension. The Proposition then follows from this.

Let $\mathscr{C}^\bullet$ be the complex (3.3.3) of overconvergent cusp forms. Consider its subcomplex formed by taking the isotypical part with $\prod_{\mathfrak{p}} U_{\mathfrak{p}}$-slope $=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_p)$. By Corollary 3.25, only the last term $(S^{\dagger}_{(\underline{k},w)})^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})}$ is nonzero. Hence, it follows from Theorem 3.5 that in the Grothendieck group of (finited generated) $\overline{\mathbb{Q}}_p[U_{\mathfrak{p}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1} : \mathfrak{p}\in\Sigma_p]$-modules, we have

(6.3.3)
$$\begin{aligned}(S^{\dagger}_{(\underline{k},w)})^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})} &= (-1)^g\big[H^\star(\mathscr{C}^\bullet)^{(\prod_{\mathfrak{p}} U_{\mathfrak{p}})\text{-slope}=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})}\big]\\ &= (-1)^g\big[H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})^{(\prod_{\mathfrak{p}} U_{\mathfrak{p}})\text{-slope}=\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})}\big].\end{aligned}$$

We need to show that the dimension of this (virtue) $\overline{\mathbb{Q}}_p$-vector space is the same as the left hand side of (6.3.2) times $(-1)^g$.

Put $N=g!$ (a very divisible number). We put $\Phi := \prod_{\mathfrak{p}\in\Sigma_p}\Phi_{\mathfrak{p}^2}^N$ with $\Phi_{\mathfrak{p}^2}=\mathrm{Fr}_{\mathfrak{p}}^2/S_{\mathfrak{p}}$. Then the slope condition above on $\prod_{\mathfrak{p}} U_{\mathfrak{p}}$ is equivalent to Φ-slope $= Nwg - 2N\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})$, by Lemma 3.22. Now we argue as in Theorem 6.1(2): fixing an isomorphisms $\overline{\mathbb{Q}}_l\cong\mathbb{C}\cong\overline{\mathbb{Q}}_p$, we have equalities in the Grothendieck group of $\overline{\mathbb{Q}}_p[\Phi]$:

$$\begin{aligned}&\big[H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})^{\Phi\text{-slope}=Nwg-2N\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})}\big]\\ \overset{(4.10.1)}{=\!=\!=}&\sum_{\mathtt{T}\subseteq\Sigma_\infty}(-1)^{-\#\mathtt{T}}\big[H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp,\mathscr{D}^{(\underline{k},w)})^{\Phi\text{-slope}=Nwg-2N\#\mathtt{T}-2N\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})}\otimes_{L_\wp}\overline{\mathbb{Q}}_p\big]\end{aligned}$$

(6.3.4)
$$\overset{\text{Prop 4.13}}{=\!=\!=\!=}\sum_{\mathtt{T}\subseteq\Sigma_\infty}(-1)^{-\#\mathtt{T}}\big[H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_l^{(\underline{k},w)})^{\Phi\text{-slope}=Nwg-2N\#\mathtt{T}-2N\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})}\big].$$

Here, we have to modify the Φ-slope starting from the first equality by $-2N\#\mathtt{T}$ in order to take account of the action of $\Phi_{\mathfrak{p}^2}$'s on Čech symbols as described in Proposition 4.10(2), which in turn comes from the commutation relation between $\Phi_{\mathfrak{p}^2}$ and Gysin isomorphisms (4.9.2).

We first claim that all terms in (6.3.4) with $\mathtt{T}\neq\emptyset$ vanishes. Note that the slope condition (6.3.1) implies that $\underline{k}\neq(2,\dots,2)$. Thus, Proposition 5.24 says that, in the Grothendieck group of $\overline{\mathbb{Q}}_l[\Phi]$-modules, we have

$$[H^{\star}_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_l^{(\underline{k},w)})]=\sum_{\pi\in\mathscr{A}_{(\underline{k},w)}}(-1)^{g-\#\mathtt{S}(\mathtt{T})_\infty}\big[(\pi^{\infty}_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}}\otimes\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}\otimes(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l(-1))^{\otimes I_{\mathtt{T}}},\big]$$

where the action of Φ on each of $\overline{\mathbb{Q}}_l(-1)$ by multiplication by p^{2N}, and the action of Φ on $\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}$ as $\rho^{\mathtt{S}(\mathtt{T})}_{\pi,l}(\mathrm{Frob}_{p^{2N}})$ times $u\cdot p^{N(w-2)(\#\mathtt{S}(\mathtt{T})_\infty-g)}$ with u a root of unity. We will show that, for each $\pi\in\mathscr{A}_{(\underline{k},w)}$, the slope of Φ on $\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}\otimes(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l(-1))^{\otimes I_{\mathtt{T}}}$ is always strictly smaller than

$$N\sum_{\tau\in\Sigma_\infty}k_\tau+2N\#I_{\mathtt{T}}-\frac{2N}{g}\min_{\tau\in\Sigma_\infty}(k_\tau-1),$$

which is easily seen to be strictly smaller than $Nwg-2N\#\mathtt{T}-2N\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})$ under our assumption (6.3.1) (and with the fact that $\#\mathtt{T}+\delta\leq\#\mathtt{T}+\#I_{\mathtt{T}}=\#\mathtt{S}(\mathtt{T})_\infty\leq g$). This would then imply that all terms in (6.3.4) with $\mathtt{T}\neq\emptyset$ is zero. Since the p-adic valuation of the number $u\cdot p^{N(w-2)(\#\mathtt{S}(\mathtt{T})_\infty-g)}$ is $(w-2)(\#\mathtt{S}(\mathtt{T})_\infty-g)N$, it remains to show that the $\rho^{\mathtt{S}(\mathtt{T})}_{\pi,l}(\mathrm{Frob}_{p^{2N}})$ has slope strictly smaller than

$$N(w-2)(g-\#\mathtt{S}(\mathtt{T})_\infty)+N\sum_{\tau\in\Sigma_\infty}k_\tau-\frac{2N}{g}\min_{\tau\in\Sigma_\infty}(k_\tau-1).\tag{6.3.5}$$

We claim that, for each $\mathfrak{p}\in\Sigma_p$ with $\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}\neq\Sigma_{\infty/\mathfrak{p}}$, the slope of $\mathrm{Frob}_{p^{2N}}$ on the unramified l-adic Galois representation $\bigotimes_{\Sigma_{\infty/\mathfrak{p}}-\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}}\text{-}\mathrm{Ind}^{\mathrm{Gal}_{\mathbb{Q}_p}}_{\mathrm{Gal}_{F_{\mathfrak{p}}}}(\rho_{\pi,l}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}})$ of $\mathrm{Gal}_{F_{\mathtt{S},\wp}}$ is less than or equal to

$$N\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}(w+k_\tau-2)(1-\frac{\#\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}}{d_{\mathfrak{p}}}).$$

Indeed, let $\alpha_{\mathfrak{p}},\beta_{\mathfrak{p}}$ denote the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ on $\rho_{\pi,l}$ with $\mathrm{val}_p(\alpha_{\mathfrak{p}})\leq\mathrm{val}_p(\beta_{\mathfrak{p}})$. Using the admissibility condition of the corresponding p-adic representation of $\mathrm{Gal}_{F_{\mathfrak{p}}}$, we have

$$\mathrm{val}_p(\beta_{\mathfrak{p}})\leq\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}\frac{w+k_\tau-2}{2}.\tag{6.3.6}$$

Therefore, the slope of $\mathrm{Frob}_{p^{2N}}$ acting on $\otimes_{\Sigma_{\infty/\mathfrak{p}}-\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}}\text{-}\mathrm{Ind}^{\mathrm{Gal}_{\mathbb{Q}_p}}_{\mathrm{Gal}_{F_{\mathfrak{p}}}}\rho_{\pi,l}$ is less than or equal to

$$2N\frac{d_{\mathfrak{p}}-\#\mathtt{S}(\mathtt{T})_{\mathfrak{p}}}{d_{\mathfrak{p}}}\mathrm{val}_p(\beta_{\mathfrak{p}})\leq N\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}(w+k_\tau-2)(1-\frac{\#\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}}{d_{\mathfrak{p}}}).$$

Note that the expression above is automatically trivial if $\mathtt{S}_{\infty/\mathfrak{p}}=\Sigma_{\infty/\mathfrak{p}}$. Hence, summing over all $\mathfrak{p}\in\Sigma_p$, we see that the eigenvalues of $\rho^{\mathtt{S}(\mathtt{T})}_{\pi,l}(\mathrm{Frob}_{p^{2N}})$ have slopes smaller than or equal to

$$N(w-2)(g-\#\mathtt{S}(\mathtt{T})_\infty)+N\sum_{\tau\in\Sigma_\infty}k_\tau-N\sum_{\mathfrak{p}\in\Sigma_p}\frac{\#\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}}{d_{\mathfrak{p}}}\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}k_\tau,$$

which is strictly smaller than (6.3.5) due to the very loose inequality

$$\sum_{\mathfrak{p}\in\Sigma_p}\frac{N\#\mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}}}{d_{\mathfrak{p}}}\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}k_\tau>\frac{2N}{g}\min_{\tau\in\Sigma_{\infty/\mathfrak{p}}}(k_\tau-1).$$

Therefore, all terms in (6.3.4) with $\mathtt{T} \neq \emptyset$ are zero. Hence, in view of (6.3.3), we get

$$\big[(S^{\dagger}_{(\underline{k},w)})^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_{\mathfrak{p}})}\big] = (-1)^g \big[H^{\star}_{c,\mathrm{et}}(X_{\overline{\mathbb{F}}_p}, \mathscr{L}_l^{(\underline{k},w)})^{\Phi\text{-slope}=Nwg-2N\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_{\mathfrak{p}})}\big],$$

where the action of Φ on the left hand side is given by $p^{2gN}(\prod_{\mathfrak{p}} S_{\mathfrak{p}}/U_{\mathfrak{p}}^2)^N$.

Similar to the argument above, Proposition 5.24 implies that (note that $\underline{k} \neq (2,\dots,2)$)

$$(6.3.7)\qquad \begin{aligned}(-1)^g\big[\quad & H^{\star}_{c,\mathrm{et}}(X_{\overline{\mathbb{F}}_p}, \mathscr{L}_l^{(\underline{k},w)})^{\Phi\text{-slope}=Nwg-2N\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_{\mathfrak{p}})}\big]\\ &= \big[\bigoplus_{\pi\in\mathscr{A}_{(\underline{k},w)}} (\pi^{\infty,p})^{K^p}\otimes \pi_p^{K_p}\otimes(\tilde{\rho}^{\emptyset}_{\pi,l})^{\Phi\text{-slope}=Nwg-2N\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_{\mathfrak{p}})}\big],\end{aligned}$$

where $\tilde{\rho}^{\emptyset}_{\pi,l} \cong \bigotimes_{\Sigma_\infty}\text{-}\mathrm{Ind}^{\mathrm{Gal}_{\mathbb{Q}}}_{\mathrm{Gal}_F}(\rho_{\pi,l})$ with Φ-action given by $\mathrm{Frob}_{p^{2N}}$ times a number $u\cdot p^{-(w-2)gN}$. In order to conclude that (6.3.2) is an isomorphism, we need to show that the right hand side above has the same dimension as

$$\big[S_{(\underline{k},w)}(K^p\mathrm{Iw}_p, \overline{\mathbb{Q}}_p)^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_p)}\big] = \big[\bigoplus_{\pi\in\mathscr{A}^{(\underline{k},w)}} (\pi^{\infty,p})^{K^p}\otimes\big(\pi_p^{\mathrm{Iw}_p}\big)^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_p)}\big].$$

It suffices to show that we have

$$(6.3.8)\qquad \dim\big(\pi_p^{K_p}\otimes(\tilde{\rho}^{\emptyset}_{\pi,l})^{\Phi\text{-slope}=Nwg-2N\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_{\mathfrak{p}})}\big) = \dim\big(\pi_p^{\mathrm{Iw}_p}\big)^{\prod_{\mathfrak{p}} U_{\mathfrak{p}}\text{-slope}=\sum_{\mathfrak{p}} \mathrm{val}_p(\lambda_p)}$$

for every $\pi\in\mathscr{A}^{(\underline{k},w)}$. Note that if π is Steinberg or supercuspidal at some places $\mathfrak{p}\in\Sigma_p$, then the both sides above equals to 0 due to the slope condition (6.3.1). Assume therefore that π is hyperspecial at p. Then $\pi_p^{K_p}$ is one-dimensional. Let $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ be the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ on $\rho_{\pi,l}$ with $\mathrm{val}_p(\alpha_{\mathfrak{p}}) \leq \mathrm{val}_p(\beta_{\mathfrak{p}})$ for any $\mathfrak{p}\in\Sigma_p$. Then $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ also coincide with eigenvalues of $U_{\mathfrak{p}}$ on $\pi_{\mathfrak{p}}^{\mathrm{Iw}_{\mathfrak{p}}}$ by Eichler-Shimura congruence relation. Hence, the $\prod_{\mathfrak{p}} U_{\mathfrak{p}}$-slopes on $\pi_p^{\mathrm{Iw}_p}$ take values in $\mathfrak{S}(U_p) := \{\sum_{\mathfrak{p}\in\Sigma_p}\mathrm{val}_p(\gamma_{\mathfrak{p}}) : \gamma_{\mathfrak{p}}\in\{\alpha_{\mathfrak{p}},\beta_{\mathfrak{p}}\}, \mathfrak{p}\in\Sigma_p\}$, and the slopes of Φ take values in the set

$$\mathfrak{S}(\Phi) := \{2N\sum_{\mathfrak{p}\in\Sigma_p}\frac{1}{d_{\mathfrak{p}}}\Big[\mathrm{val}_p(\alpha_{\mathfrak{p}})i_{\mathfrak{p}} + (d_{\mathfrak{p}}-i_{\mathfrak{p}})\mathrm{val}_{\mathfrak{p}}(\beta_{\mathfrak{p}})\Big] - Ng(w-2) : i_{\mathfrak{p}}\in\mathbb{Z}\cap[0,d_{\mathfrak{p}}]\}.$$

Then if $\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})$ is not in the set

$$\widetilde{\mathfrak{S}}(U_p) := \mathfrak{S}(U_p)\cup\{\sum_{\mathfrak{p}}\mathrm{val}_p(\alpha_{\mathfrak{p}}) + \sum_{\mathfrak{p}}\frac{i_{\mathfrak{p}}}{d_{\mathfrak{p}}}(\mathrm{val}_p(\beta_{\mathfrak{p}}) - \mathrm{val}_p(\alpha_{\mathfrak{p}})) : i_{\mathfrak{p}}\in\mathbb{Z}\cap[0,d_{\mathfrak{p}}], \forall\mathfrak{p}\in\Sigma_p\}$$

then both sides of (6.3.7) are equal to 0. Assume therefore $\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})\in\tilde{\mathfrak{S}}(U_p)$ and (6.3.1) is satisfied. Note that $\mathrm{val}_p(\alpha_{\mathfrak{p}}) < \mathrm{val}_p(\beta_{\mathfrak{p}})$ for all $\mathfrak{p}\in\Sigma_p$, since $\mathrm{val}_p(\alpha_{\mathfrak{p}})+\mathrm{val}_p(\beta_{\mathfrak{p}}) = (w-1)d_{\mathfrak{p}}$. We claim that $\mathrm{val}_p(\lambda_{\mathfrak{p}}) = \sum_{\mathfrak{p}}\mathrm{val}_p(\alpha_{\mathfrak{p}})$, which is the minimal element of $\widetilde{\mathfrak{S}}(U_p)$. It follows that both sides of (6.3.7) have dimension 1, and the proof will be finished. To prove the claim, it suffices to show that

$$\sum_{\mathfrak{p}\in\Sigma_p}\mathrm{val}_p(\alpha_{\mathfrak{p}}) + \min_{\mathfrak{p}}\frac{1}{d_{\mathfrak{p}}}(\mathrm{val}_p(\beta_{\mathfrak{p}}) - \mathrm{val}_p(\alpha_{\mathfrak{p}})) > \sum_{\tau\in\Sigma_\infty}\frac{w-k_\tau}{2} + \frac{1}{g}\min_{\tau}(k_\tau-1) - g$$

where the right hand side is greater than $\sum_{\mathfrak{p}}\mathrm{val}_p(\lambda_{\mathfrak{p}})$ by assumption. Let $\mathfrak{p}_0\in\Sigma_p$ so that the minimal of the left hand side is achieved. Since $\mathrm{val}_p(\alpha_{\mathfrak{p}}) \geq \sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}\frac{w-k_\tau}{2}$ by admissibility condition and $\mathrm{val}_p(\alpha_{\mathfrak{p}_0}) + \mathrm{val}_p(\beta_{\mathfrak{p}_0}) = d_{\mathfrak{p}_0}(w-1)$, we are easily reduced to showing that

$$\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}_0}}\frac{k_\tau-1}{2} - (\frac{1}{2}-\frac{1}{d_{\mathfrak{p}_0}})[\mathrm{val}_p(\beta_{\mathfrak{p}_0}) - \mathrm{val}_p(\alpha_{\mathfrak{p}_0})] > \frac{1}{g}\min_{\tau\in\Sigma_\infty}(k_\tau-1) - g,$$

which is trivially true if $d_{\mathfrak{p}_0}\leq 2$, and follows easily from $\mathrm{val}_p(\beta_{\mathfrak{p}_0}) - \mathrm{val}_p(\alpha_{\mathfrak{p}_0}) \leq \sum_{\tau\in\Sigma_{\infty/\mathfrak{p}_0}}(k_\tau-1)$ if $d_{\mathfrak{p}_0} > 2$. $\square$

6.4. Overconvergent Eigenforms of level $K_1(\mathfrak{N})$. — Let $\mathfrak{N}$ be an integral ideal of $\mathcal{O}_F$ prime to p. We put

$$K_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) | a \equiv 1, c \equiv 0 \mod \mathfrak{N} \right\},$$

and let $K_1(\mathfrak{N})^p$ be the prime-to-p part. Since $K_1(\mathfrak{N})^p$ does not satisfy Hypothesis 2.7, the theory in Section 3 does not apply directly. By Lemma 2.5, we choose an open compact subgroup $K^p \subseteq K_1(\mathfrak{N})^p$ that satisfies Hypothesis 2.7. Consider the space $S^{\dagger}_{(\underline{k},w)}(K, L_{\wp})$ of overconvergent cusp forms of level $K = K^pK_p$ for a sufficiently large finite extension $L_{\wp}/\mathbb{Q}_p$. We have a natural action of $\Gamma := K_1(\mathfrak{N})^p/K^p$ on $S^{\dagger}_{\underline{k},w}(K, L_{\wp})$. We define the space of overconvergent cusp forms of level $K_1(\mathfrak{N})$ to be the invariant subspace

$$S^{\dagger}_{(\underline{k},w)}(K_1(\mathfrak{N}), L_{\wp}) := S^{\dagger}_{(\underline{k},w)}(K, L_{\wp})^{\Gamma}.$$

It is easy to see that the definition does not depend on the choice of K^p. The Hecke algebra $\mathscr{H}(K_1(\mathfrak{N})^p, L_{\wp})$ and the operators $U_{\mathfrak{p}}, S_{\mathfrak{p}}$ for $\mathfrak{p} \in \Sigma_p$ acts naturally on $S^{\dagger}_{(\underline{k},w)}(K_1(\mathfrak{N}), L_{\wp})$. We say $f \in S^{\dagger}_{(\underline{k},w)}(K_1(\mathfrak{N}), L_{\wp})$ is a *(normalized) overconvergent eigenform* if f is a common eigenvector for all the Hecke operators in $\mathscr{H}(K_1(\mathfrak{N})^p, L_{\wp})[U_{\mathfrak{p}}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1} : \mathfrak{p} \in \Sigma_p]$, and the first Fourier coefficient (the coefficient indexed by $1 \in \mathcal{O}_F$) is 1. Note that $\mathscr{H}(K_1(\mathfrak{N})^p, L_{\wp})$ is generated by the usual Hecke operators T_v for $v \nmid p\mathfrak{N}$, U_v for $v|\mathfrak{N}$, together with S_v and S_v^{-1} for all $v \nmid p$.

Let $\widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$ denote the subspace of classical Hilbert modular forms which vanishes at unramified cusps of the Hilbert modular variety of level $K_1(\mathfrak{N})^p\mathrm{Iw}_p$. There are natural Hecke equivariant injections

$$S_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp}) \hookrightarrow \widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp}) \hookrightarrow S^{\dagger}_{(\underline{k},w)}(K_1(\mathfrak{N}), L_{\wp}),$$

where $S_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$ is the space of the classical Hilbert cusp forms. We will say a form $f \in S^{\dagger}_{(\underline{k},w)}(K_1(\mathfrak{N}), L_{\wp})$ is a *classical Hilbert modular form* if it lies in $\widetilde{S}^{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$, and is a *classical Hilbert cusp form* if it is in $S_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$.

Note that if $(\underline{k}, w)$ is not of parallel weight, then $\widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$ coincides with $S_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$; but if $(\underline{k}, w)$ is of parallel weight k, then $\widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$ will contain as well some Eisenstein series of level $K_1(\mathfrak{N})^p\mathrm{Iw}_p$ of $U_{\mathfrak{p}}$-slope $d_{\mathfrak{p}}(k-1)$ for all $\mathfrak{p} \in \Sigma_p$. Actually, let χ be an algebraic Hecke character of F with archimedean component given by $N_{F/\mathbb{Q}}^{k-2}$ and of conductor $\mathfrak{c}$ dividing $\mathfrak{N}$. Write $\chi = \epsilon|\cdot|^{k-2}$ with ϵ a finite Hecke character. Then there exists an Eistenstein series E_{χ} of weight k such that E_{χ} is a common eigenvector of $T_{\mathfrak{q}}$ with eigenvalue $1 + \epsilon(\mathfrak{q}^{-1})N_{F/\mathbb{Q}}(\mathfrak{q})^{k-1}$ for all prime ideals $\mathfrak{q} \nmid \mathfrak{c}$. We take the p-stablization E'_{χ} of E_{χ} such that E'_{χ} has level $K_1(\mathfrak{N})^p\mathrm{Iw}_p$ and it is a common eigenvector of $U_{\mathfrak{p}}$ with eigenvalue $\epsilon(\mathfrak{p}^{-1})N_{F/\mathbb{Q}}(\mathfrak{p})^{w-1}$ for all $\mathfrak{p} \in \Sigma_p$. Then E'_{χ} vanishes at all unramified cusps at p of the Hilbert modular variety of level $K_1(\mathfrak{N})^p\mathrm{Iw}_p$, and hence is contained in $\widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$.

Recall that Kisin and Lai constructed in [**KL05**] various eigencurves $\mathcal{C}_{(\underline{k},w)}(\bar{\rho})$ that parametrize (normalized) overconvergent eigenforms with different weights. The points on the Kisin-Lai eigencurves that correspond to classical Hilbert eigenforms are called classical points.

Theorem 6.5. — *On the Kisin-Lai eigencurves for overconvergent cusp forms, classical points are Zariski dense.*

Proof. — This follows immediately from Proposition 6.3 by the same arguments as [**Ti11**, Theorem 2.20]. □

Remark 6.6. — This Theorem also follows of course from the main results of [**PS11**], where Pilloni and Stroh proved the classicality of overconvergent Hilbert modular forms using the method of analytic continuation.

The following is the combination of the work of many people.

Proposition 6.7. — *Let $f \in S^{\dagger}_{(\underline{k},w)}(K_1(\mathfrak{N}), L_{\wp})$ be an overconvergent eigenform. Then there exists a p-adic Galois representation ρ_f of G_F attached to f such that the following properties are satisfied:*

- *For every finite place $\mathfrak{l} \nmid p\mathfrak{N}$, if $\lambda_{\mathfrak{l}}$ denotes the eigenvalue of the Hecke operator $T_{\mathfrak{l}}$ on f, then ρ_f is unramified at $\mathfrak{l}$ and the characteristic polynomial of $\rho_f(\mathrm{Frob}_{\mathfrak{l}})$ is $T^2 - \lambda_{\mathfrak{l}} T + \epsilon(\mathfrak{l}) N_{F/\mathbb{Q}}\mathfrak{l}^{(w-1)}$, where $\epsilon(\mathfrak{l})$ is a root of unity, and equals to the eigenvalue of $S_{\mathfrak{l}}$ on f divided by $N_{F/\mathbb{Q}}(\mathfrak{l})^{(w-2)}$.*
- *For a place $\mathfrak{p} \in \Sigma_p$, if $\lambda_{\mathfrak{p}}$ is the eigenvalue of the $U_{\mathfrak{p}}$-operator, then the local Galois representation $\rho_f|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is Hodge-Tate with Hodge-Tate weights $\frac{w-k_\tau}{2}, \frac{w+k_\tau-2}{2}$ in the τ-direction for each $\tau \in \Sigma_{\infty/\mathfrak{p}}$, $\mathbb{D}_{\mathrm{cris}}(\rho_f|_{\mathrm{Gal}_{F_{\mathfrak{p}}}})^{\mathrm{Frob}_{\mathfrak{p}}=\lambda_{\mathfrak{p}}}$ is nonzero and its image in $\mathbb{D}_{\mathrm{dR},\tau}(\rho_f|_{\mathrm{Gal}_{F_{\mathfrak{p}}}})$ lies in $\mathrm{Fil}^{(w-k_\tau)/2}\mathbb{D}_{\mathrm{dR},\tau}(\rho_f|_{\mathrm{Gal}_F})$.*
- *If f is classical, then ρ_f is semistable (including crystalline) at all places $\mathfrak{p} \in \Sigma_p$.*

Proof. — When f is classical, the construction of ρ_f is due to Carayol [**Ca86b**], Taylor [**Ta89**] and Blasuis-Rogowski [**BR93**]. The verification of the properties for ρ_f was done by Carayol [**Ca86b**] for places outside p and by Saito [**Sa09**] (plus a special case handled independently by Tong Liu [**Li12**] and Christopher Skinner [**Sk09**]) at places above p. For a general f, we consider an Kisin-Lai eigencurve $\mathcal{C}$ that passes through f. Then the continuity of the Hecke eigenvalues define a pseudo-representations over the reduced subscheme of $\mathcal{C}$. Specializing this pseudo-representation to the point corresponding to f provides f with a Galois representation of Gal_F.

The existence of crystalline periods can be proved using the recent work of Kedlaya, Pottharst and the second author [**KPX12**], or independently R. Liu [**Li12+**] on global triangulation. Both works generalize prior work of Kisin [**Ki12**]. □

Corollary 6.8. — *Let $f \in S^{\dagger}_{(\underline{k},w)}(K_1(\mathfrak{N}), L_{\wp})$ be an overconvergent eigenform. Assume that there exists a classical eigenform $\tilde{f} \in \widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$ such that f and $\tilde{f}$ have the same eigensystem for $\mathscr{H}(K_1(\mathfrak{N})^p, L_{\wp})$. Then f lies in $\widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_{\wp})$.*

Proof. — Let $\pi_{\tilde{f}}$ be the automorphic representation generated by $\tilde{f}$. Then $\pi_{\tilde{f}}$ has conductor $\mathfrak{c}$ dividing $p\mathfrak{N}$. We denote by $\Delta_{\tilde{f}}(\mathfrak{N})$ the set of $K_1(\mathfrak{N})^p\mathrm{Iw}_p$-eigenforms contained in $\pi_{\tilde{f}}$, i.e. the set of the various $\mathfrak{q}$-stablizations of the newform in $\pi_{\tilde{f}}$ with $\mathfrak{q}$ dividing $p\mathfrak{N}/\mathfrak{c}$. Since f and $\tilde{f}$ have the same tame Hecke eigensystem, the p-adic Galois representation ρ_f is isomorphic to $\rho_{\tilde{f}}$ (or more canonically $\rho_{\pi_{\tilde{f}}}$) by Chebotarev density. In particular, $\mathbb{D}_{\mathrm{cris}}(\rho_f|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}) \simeq \mathbb{D}_{\mathrm{cris}}(\rho_{\tilde{f}}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}})$ for every $\mathfrak{p} \in \Sigma_p$. If $\lambda_{\mathfrak{p}}(f)$ denotes the eigenvalue of $U_{\mathfrak{p}}$ on f, then $\lambda_{\mathfrak{p}}(f)$ appears as an eigenvalue of $\mathrm{Frob}_{\mathfrak{p}}$ on $\mathbb{D}_{\mathrm{cris}}(\rho_{\tilde{f}}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}})$ by Proposition 6.7. Then there exists $\tilde{f}' \in \Delta_{\tilde{f}}$ such that $\lambda_{\mathfrak{p}}(f) = \lambda_{\mathfrak{p}}(\tilde{f}')$ for all primes $\mathfrak{p} \in \Sigma_p$. We conclude by the q-expansion principle that $f = \tilde{f}'$. □

Theorem 6.9 **(Strong classicality)**. — *Let f be a cuspidal overconvergent Hilbert eigenform of multiweight $(\underline{k}, w)$ of level $K = K^pK_p$ with $K_p = \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$. Let*

$$d^{g-1} : \mathscr{C}^{g-1} = \bigoplus_{\tau \in \Sigma_\infty} S^{\dagger}_{\epsilon_{\Sigma_\infty - \{\tau\}} \cdot (\underline{k},w)}(K, L_{\wp}) \xrightarrow{\sum_{\tau \in \Sigma_\infty} \Theta_{\tau, k_\tau - 1}} \mathscr{C}^g = S^{\dagger}_{(\underline{k},w)}(K, L_{\wp})$$

denote the $(g-1)$-th differential morphism of the complex $\mathscr{C}^\bullet$ (3.3.3).

1. *If f is not in the image of d^{g-1}, then f lies in $\widetilde{S}_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_\wp)$.*
2. *For each $\mathfrak{p}\in\Sigma_p$, let $\lambda_\mathfrak{p}$ denote the eigenvalue of f for the operator $U_\mathfrak{p}$. If*

$$\mathrm{val}_p(\lambda_\mathfrak{p}) < \sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}} \frac{w-k_\tau}{2} + \min_{\tau\in\Sigma_{\infty/\mathfrak{p}}}\{k_\tau - 1\} \tag{6.9.1}$$

for each $\mathfrak{p}\in\Sigma_\mathfrak{p}$, then f is lies in $S_{(\underline{k},w)}(K_1(\mathfrak{N})^p\mathrm{Iw}_p, L_\wp)$.

Proof. — (1) By Corollary 6.8, it suffices to prove the tame Hecke eigenvalues of f coincide with those of a classical cuspidal eigenform. By Theorem 3.5, f gives rise to a non-zero cohomology class in $H^g_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$. It follows then from Theorem 6.1(1) that the tame Hecke eigenvalues of f come from an automorphic representation π of $\mathrm{GL}_{2,F}$ whose central character is an algebraic Hecke character with archimidean part $N_{F/\mathbb{Q}}^{w-2}$. Such a π might be cuspidal, one-dimensional or Eisenstein. We have to exclude the case of one-dimensional representation, and then (1) would follow from Corollary 6.8. Assume in contrary that f has the same tame Hecke spectrum as a one-dimensional automorphic representation:

$$\mathrm{GL}_2(\mathbb{A}_F) \xrightarrow{\det} \mathbb{A}_F^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

Then χ is an algebraic Hecke character whose restriction to $(F\otimes\mathbb{R})^{\times,\circ}$ is $N_{F/\mathbb{Q}}^{w/2-1}$. Via the fixed isomorphism $\iota_p : \mathbb{C}\simeq\overline{\mathbb{Q}}_p$, we have a well defined p-adic character on $\mathbb{A}_F^\times$

$$\chi_p : x\mapsto (\chi(x)\cdot N_{F/\mathbb{Q}}(x_\infty)^{1-\frac{w}{2}})N_{F/\mathbb{Q}}(x_p)^{\frac{w}{2}-1}\in\overline{\mathbb{Q}}_p^\times$$

where x_∞ and x_p are respectively the archimedean and p-adic component of x. Note that χ_p is trivial on $(F\otimes\mathbb{R})^{\times,\circ}\cdot F^\times$. By class field theory, it defines a p-adic character of Gal_F. We put $\rho_{\pi,p}=\chi_p^{-1}$. Then $\rho_{\pi,p}$ is a one-dimensional Galois representation of Gal_F such that if $\mathfrak{l}\nmid p\mathfrak{N}$ is a place of F, $\mathrm{Tr}(\rho_{\pi,p}(\mathrm{Frob}_\mathfrak{l}))$ equals to the eigenvalue of the Hecke operator $T_\mathfrak{l}$ on f. However, Proposition 6.7 implies that there is a two dimensional p-adic representation ρ_f of Gal_F satisfying the same property. Hence, we have $\mathrm{Tr}(\rho_{\pi,p}(\mathrm{Frob}_\mathfrak{l}))=\mathrm{Tr}(\rho_f(\mathrm{Frob}_\mathfrak{l}))$ for all unramified primes $\mathfrak{l}$. By Chebotarev density, this implies that the semi-simplification of ρ_f is equal to $\rho_{\pi,p}$, which is absurd. This finishes the proof of the first part.

(2) If f satisfies the condition (6.9.1), then Proposition 3.25 implies that f is not in the image of d^{g-1}. Hence, f is a classical (cuspidal) Hilbert eigenform by the first part of the Theorem. Such an f can not be Eisenstein either, because an Eisenstein series of (parallel) weight $(\underline{k},w)$ appearing in $H^g_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ must have $\prod_{\mathfrak{p}\in\Sigma_p} U_\mathfrak{p}$-slope $(w-1)g$. This shows that f must be a classical cuspidal Hilbert modular form. □

7. Computation of the Rigid Cohomology II

We keep the notation of the previous section. If we assume that Conjecture 5.18 holds (e.g. when p is inert in F by Theorem 5.12), we can strengthen the weak cohomology computation Theorem 6.1 to a stronger version by including the action of $U_\mathfrak{p}^2$'s.

***Theorem 7.1* (Strong cohomology computation).** — *Assume Conjecture 5.18. We have the following isomorphism of modules in the Grothendieck group of modules of $\mathscr{H}(K^p, \overline{L}_\wp)[S_\mathfrak{p}, S_\mathfrak{p}^{-1}, U_\mathfrak{p}^2; \mathfrak{p}\in\Sigma_p]$.*

$$\big[H^*_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})\big] = (-1)^g\cdot\big[S_{(\underline{k},w)}(K^p\mathrm{Iw}_p, L_\wp)\big]. \tag{7.1.1}$$

Before giving the proof of this theorem, we deduce a corollary which slightly strengthens Theorem 6.9 on classicality of overconvergent cusp forms.

Corollary 7.2. — *Assume Conjecture 5.18. Let $(\underline{k},w)$ be a multiweight. For every $\mathfrak{p}\in\Sigma_p$, we put $s_\mathfrak{p}=\sum_{\tau\in\Sigma_{\infty/\mathfrak{p}}}\frac{w-k_\tau}{2}+\min_{\tau\in\Sigma_{\infty/\mathfrak{p}}}\{k_\tau-1\}$. Then the natural injection*

$$S_{(\underline{k},w)}(K^p\mathrm{Iw}_p, L_\wp)^{U_\mathfrak{p}\text{-}slope<s_\mathfrak{p}} \hookrightarrow S^\dagger_{(\underline{k},w)}(K, L_\wp)^{U_\mathfrak{p}\text{-}slope<s_\mathfrak{p}}$$

is an isomorphism, where the superscript means taking the part of $U_{\mathfrak{p}}$-slope strictly less than $s_{\mathfrak{p}}$ for every $\mathfrak{p} \in \Sigma_p$. In particular, if f is an overconvergent cusp form of multiweight $(\underline{k}, w)$ and level K^pK_p with $K_p = \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$, which is a generalized eigenvector of $U_{\mathfrak{p}}$ with slope strictly less than $s_{\mathfrak{p}}$ for every $\mathfrak{p} \in \Sigma_p$, then f is classical.

Proof. — It suffices to compare the dimensions. First of all, Theorem 3.5 implies that

$$(\mathscr{C}^\bullet)^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}} \cong \big(\mathrm{R}\Gamma_{\mathrm{rig}}(Y^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})\big)^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}}. \tag{7.2.1}$$

By the slope inequality (Proposition 3.25), the left hand side of (7.2.1) has only one term $(S^\dagger_{(\underline{k},w)})^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}}$ in degree g. So the right hand side of (7.2.1) is also concentrated in degree g. By Theorem 7.1 above, the right hand side of (7.2.1) in the Grothendieck group of modules of $L_\wp[U_{\mathfrak{p}}^2; \mathfrak{p} \in \Sigma_p]$ equals to $(-1)^g \cdot \big[S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}}\big]$. It follows that $[(S^\dagger_{(\underline{k},w)})^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}}] = [S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}}]$, and in particular

$$\dim(S^\dagger_{(\underline{k},w)})^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}} = \dim S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}}.$$

Therefore, the natural inclusion $S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}} \hookrightarrow S^\dagger_{(\underline{k},w)}(K^pK_p)^{U_{\mathfrak{p}}\text{-slope}<s_{\mathfrak{p}}}$ must be an isomorphism. □

The proof of Theorem 7.1 will occupy rest of the paper. We start by breaking up the theorem into a local computation at each place $\mathfrak{p} \in \Sigma_p$.

7.3. Reduction of the proof of Theorem 7.1. — Recall that we have introduced Čech symbols e_τ and $e_{\mathtt{T}}$ in 4.2 and 4.10 to encode the action of $\mathrm{Fr}_{\mathfrak{p}}$ and $\Phi_{\mathfrak{p}}$. We need more Čech symbols for the second relative cohomology of the $\mathbb{P}^1$-bundle and the $\Phi_{\mathfrak{p}^2}$-action on them. For each $\tau \in \Sigma_\infty$, we introduce a *Čech symbol* f_τ of degree 2. This means that $f_\tau \wedge f_{\tau'} = f_\tau \wedge f_{\tau'}$ and $f_\tau \wedge e_{\tau'} = e_{\tau'} \wedge f_\tau$ for all $\tau, \tau' \in \Sigma_\infty$. For a subset $\mathtt{T} \subseteq \Sigma_\infty$, we put $f_{\mathtt{T}} = \bigwedge_{\tau \in \mathtt{T}} f_\tau$. The action of $\Phi_{\mathfrak{p}^2}$ on the formal Čech symbols is defined to be

$$\Phi_{\mathfrak{p}^2}(e_\tau) = p^2 e_{\sigma^{-2}\tau} \quad \text{and} \quad \Phi_{\mathfrak{p}^2}(f_\tau) = p^2 f_{\sigma^{-2}\tau},$$

for $\tau \in \Sigma_{\infty/\mathfrak{p}}$, and is trivial on e_τ and f_τ for $\tau \notin \Sigma_{\infty/\mathfrak{p}}$. For a subset $\mathtt{T} \subset \Sigma_\infty$, let $\mathtt{S}(\mathtt{T})$ be the subset defined in Subsection 5.21, and put $I_{\mathtt{T}} = \mathtt{S}(\mathtt{T})_\infty - \mathtt{T}$ and $I_{\mathtt{T},\mathfrak{p}} = I_{\mathtt{T}} \cap \Sigma_{\infty/\mathfrak{p}}$. We fix isomorphisms $\overline{\mathbb{Q}}_p \cong \mathbb{C} \cong \overline{\mathbb{Q}}_l$ as usual. Under the assumption of Conjecture 5.18, we have a more delicate computation than Theorem 6.1 in the Grothendieck group of $\mathscr{H}(K^p, \overline{\mathbb{Q}}_p)[\Phi_{\mathfrak{p}^2}; \mathfrak{p} \in \Sigma_p]$:

$$
\begin{aligned}
&(-1)^g\big[H^\star_{\mathrm{rig}}(Y^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})\otimes_{L_\wp}\overline{\mathbb{Q}}_p\big]\\
\overset{(4.10.1)}{=\!=}&\sum_{i=0}^{g}(-1)^{g-i}\Big[\bigoplus_{\#\mathtt{T}=i}H^\star_{c,\mathrm{rig}}(Y_{\mathtt{T}}/L_\wp,\mathscr{D}^{(\underline{k},w)})\otimes_{L_\wp}\overline{\mathbb{Q}}_p e_{\mathtt{T}}\Big]\\
\overset{\text{Prop 4.13}}{=\!=}&\sum_{i=0}^{g}(-1)^{g-i}\Big[\bigoplus_{\#\mathtt{T}=i}H^\star_{c,\mathrm{et}}(Y_{\mathtt{T},\overline{\mathbb{F}}_p},\mathscr{L}_l^{(\underline{k},w)})e_{\mathtt{T}}\Big]\\
\overset{\text{Cor 5.24}}{=\!=}&\sum_{i=0}^{g}(-1)^{g-i}\sum_{j=0}^{i}(-1)^{g-i-j}\Big[\bigoplus_{\substack{\#\mathtt{T}=i,\\ \text{s.t.}\#I_{\mathtt{T}}=j}}e_{\mathtt{T}}\bigoplus_{\pi\in\mathscr{A}_{(\underline{k},w)}}(\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}}\otimes\tilde{\rho}^{\mathtt{S}(\mathtt{T})}_{\pi,l}\otimes\bigotimes_{\tau\in I_{\mathtt{T}}}(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_lf_\tau)\Big]\\
&+\sum_{i=0}^{g}(-1)^{g-i}\delta_{\underline{k},2}\Big[\bigoplus_{\#\mathtt{T}=i}e_{\mathtt{T}}\bigoplus_{\pi\in\mathscr{B}^w}(\pi^\infty_{\mathtt{S}(\mathtt{T})})^{K_{\mathtt{S}(\mathtt{T})}}\otimes\Big(\bigotimes_{\tau\in\Sigma_\infty-\mathtt{T}}(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_lf_\tau)-\delta_{\mathtt{T},\emptyset}\overline{\mathbb{Q}}_l\Big)\Big]\Big)\\
=&\sum_{\pi\in\mathscr{A}_{(\underline{k},w)}}\big[(\pi^{\infty,p})^{K^p}\big]\prod_{\mathfrak{p}\in\Sigma_p}\sum_{i_{\mathfrak{p}}=0}^{d_{\mathfrak{p}}}\sum_{j_{\mathfrak{p}}=0}^{i_{\mathfrak{p}}}(-1)^{j_{\mathfrak{p}}}\Big[\bigoplus_{\substack{\#\mathtt{T}_{/\mathfrak{p}}=i_{\mathfrak{p}},\\ \text{s.t.}\#I_{\mathtt{T},\mathfrak{p}}=j_{\mathfrak{p}}}}e_{\mathtt{T}_{/\mathfrak{p}}}\tilde{\rho}^{\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}}_{\pi,\mathfrak{p},l}\otimes(\pi_{\mathtt{S}(\mathtt{T}),\mathfrak{p}})^{K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}}\otimes\bigotimes_{\tau\in I_{\mathtt{T},\mathfrak{p}}}(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_lf_\tau)\Big]\\
&+\delta_{\underline{k},2}(-1)^g\sum_{\pi\in\mathscr{B}^w}\big[(\pi^\infty)^K\big]\Big(-[\overline{\mathbb{Q}}_l]+\prod_{\mathfrak{p}\in\Sigma_p}\sum_{i_{\mathfrak{p}}=0}^{d_{\mathfrak{p}}}(-1)^{i_{\mathfrak{p}}}\Big[\bigoplus_{\#\mathtt{T}_{/\mathfrak{p}}=i_{\mathfrak{p}}}e_{\mathtt{T}_{/\mathfrak{p}}}\bigotimes_{\tau\in\Sigma_{\infty/\mathfrak{p}}-\mathtt{T}_{/\mathfrak{p}}}(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_lf_\tau)\Big]\Big)
\end{aligned}
$$

where $\tilde{\rho}^{\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}}_{\pi,\mathfrak{p},l}$ is isomorphic to $\bigotimes_{\Sigma_{\infty/\mathfrak{p}-\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}}}\text{-}\mathrm{Ind}^{\mathrm{Gal}_{\mathbb{Q}_p}}_{\mathrm{Gal}_{F_{\mathfrak{p}}}}\rho_{\pi,l}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ but it carries an action of $\Phi^{n_{\mathfrak{p}}}_{\mathfrak{p}^2}$, which is the same as the action of $\mathrm{Frob}_{p^{2n_{\mathfrak{p}}}}$ on the tensorial induction representation multiplied with the number $\omega_\pi(\varpi_{\mathfrak{p}})^{n_{\mathfrak{p}}(d_{\mathfrak{p}}-\#\mathtt{S}(\mathtt{T})_{/\mathfrak{p}})/d_{\mathfrak{p}}}$. Here $n_{\mathfrak{p}}$ denotes the minimal number such that $\sigma_{\mathfrak{p}}^{n_{\mathfrak{p}}}\mathtt{T}_{/\mathfrak{p}}=\mathtt{T}_{/\mathfrak{p}}$.

We will show that the long expression above equals to

$$\sum_{\pi\in\mathscr{A}_{(\underline{k},w)}}\big[(\pi^\infty)^{K^p\mathrm{Iw}_p}\big]=\sum_{\pi\in\mathscr{A}_{(\underline{k},w)}}\big[(\pi^{\infty,p})^{K^p}\big]\prod_{\mathfrak{p}\in\Sigma_p}\big[(\pi_{\mathfrak{p}})^{\mathrm{Iw}_{\mathfrak{p}}}\big].$$

For this, we need only to discuss separately for each $\pi\in\mathscr{A}_{(\underline{k},w)}$ and $\mathscr{B}^w$. We start with the latter where the computation is much easier.

7.4. Contribution of the one-dimensional representations. — We fix $\pi\in\mathscr{B}^w$. By the discussion in the Subsection above, we need to show that the contribution of the π-component to $\big[H^\star_{\mathrm{rig}}(Y^{\mathrm{tor,ord}},\mathsf{D};\mathscr{F}^{(\underline{k},w)})\big]$ is trivial. For this, it is enough to show that, for each $\mathfrak{p}\in\Sigma_p$, we have, in the Grothendieck group of $\overline{\mathbb{Q}}_l[\Phi_{\mathfrak{p}}]$,

$$\sum_{i_{\mathfrak{p}}=0}^{d_{\mathfrak{p}}}(-1)^{i_{\mathfrak{p}}}\Big[\bigoplus_{\#\mathtt{T}_{/\mathfrak{p}}=i_{\mathfrak{p}}}e_{\mathtt{T}_{/\mathfrak{p}}}\bigotimes_{\tau\in\Sigma_{\infty/\mathfrak{p}}-\mathtt{T}_{/\mathfrak{p}}}(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_lf_\tau)\Big]\tag{7.4.1}$$

is equal to $[\overline{\mathbb{Q}}_l]$, where $\Phi_{\mathfrak{p}}$ acts on $\overline{\mathbb{Q}}_l$ trivially and

$$\Phi_{\mathfrak{p}}(e_\tau)=pe_{\sigma_{\mathfrak{p}}^{-1}\tau},\quad\text{and}\quad\Phi_{\mathfrak{p}}(f_\tau)=pf_{\sigma_{\mathfrak{p}}^{-1}\tau}.$$

(We get back to our original statement by matching the action of $\Phi_{\mathfrak{p}^2}$ with $\Phi_{\mathfrak{p}}^2$.) Since the argument will be independent for each place; we will suppress the subscript $\mathfrak{p}$ for the rest of this subsection. We also label e_τ and f_τ's as $e_1,\dots,e_d$ and $f_1,\dots,f_d$ with the convention that the subscripts are taken modulo d and $\Phi(e_i)=pe_{i-1},\Phi(f_i)=pf_{i-1}$.

This will follow from some abstract nonsense of tensorial induction. Recall the setup in Subsection 5.9: G a group, H a subgroup of finite index, $\Sigma\subseteq G/H$ a finite subset, and H' the stabilizer group of Σ under the action of G on G/H. We fix representatives $s_1,\dots,s_r$ of G/H

and assume that $\Sigma = \{s_1H, \dots, s_rH\}$ for some r. Instead of starting with a representation of H, we start with a bounded complex $C^\bullet$ of $\mathbb{Z}[H]$-modules. The *tensorial induced complex*, denoted by $\otimes_\Sigma\text{-}\operatorname{Ind}_H^G C^\bullet$, is defined to be $\otimes_{i=1}^r C_i^\bullet$, where $C_i^\bullet$ is a copy of $C^\bullet$. The action of H' on this complex is given as follows: for each $h' \in H'$, there exists a permutation j of r letters (depending on h') such that for each $i \in \{1, \dots, r\}$, $j(i) \in \{1, \dots, r\}$ is the unique element with $h's_{j(i)} \in s_iH$. We define the action of h' on $\otimes_\Sigma\text{-}\operatorname{Ind}_H^G C^\bullet$ by the linear combination of

$$\begin{aligned} h' \colon C_1^{a_1} \otimes \cdots \otimes C_r^{a_r} &\longrightarrow C_1^{a_{j(1)}} \otimes \cdots \otimes C_r^{a_{j(r)}} \\ h'(v_1 \otimes \cdots \otimes v_r) &\longmapsto \operatorname{sgn}(j) \cdot (s_1^{-1}h's_{j(1)})(v_{j(1)}) \otimes \cdots \otimes (s_r^{-1}h's_{j(r)})(v_{j(r)}). \end{aligned}$$

Note that the sign function for j is inserted to account for the sign convention in the definition of tensor product of complexes. Such a construction is functorial in $C^\bullet$ and it takes quasi-isomorphic complexes to quasi-isomorphic complexes (this is because when checking quasi-isomorphism one can forget about the group action.)

We will consider a particular case of tensorial induction: $G = \mathbb{Z}$, $H = d\mathbb{Z}$, $\Sigma = G/H = \mathbb{Z}/d\mathbb{Z}$, and the complex is given by $\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l f \to \overline{\mathbb{Q}}_l e$ in degrees 0 and 1, where the element $d \in H$ acts trivially on $\overline{\mathbb{Q}}_l$ and acts by multiplication by $1/p$ on both e and f, and the map is given by sending the first copy of $\overline{\mathbb{Q}}_l$ to zero and sending f to e. It is clear that we have a quasi-isomorphism

$$\overline{\mathbb{Q}}_l \cong \otimes_\Sigma\text{-}\operatorname{Ind}_H^G \overline{\mathbb{Q}}_l \xrightarrow{\text{quasi-isom}} \otimes_\Sigma\text{-}\operatorname{Ind}_H^G \big(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l f \twoheadrightarrow \overline{\mathbb{Q}}_l e\big).$$

The upshot is that if we think of the action of $-1 \in G$ is the action of Φ from (7.4.1), the expression (7.4.1) is exactly the image of the complex $\otimes_\Sigma\text{-}\operatorname{Ind}_H^G \big(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l f \twoheadrightarrow \overline{\mathbb{Q}}_l e\big)$ in the Grothendieck group of $\overline{\mathbb{Q}}_l[\Phi]$. Then Theorem 7.1 for one-dimensional π would follow from this.

We now explain carefully the construction of the tensorial induction for $\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l f \twoheadrightarrow \overline{\mathbb{Q}}_l e$. It is first of all isomorphic to $\otimes_{i=1}^d \big(\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l f_i \twoheadrightarrow \overline{\mathbb{Q}}_l e_i\big)$. To properly account for the sign involved in tensorial induction, we need to declare that e_i has degree 1 as a Čech symbol. The statement is now clear.

7.5. Contribution of the cuspidal representations. — To prove the final statement in Subsection 7.3 for a representation $\pi \in \mathscr{A}_{(\underline{k},w)}$, we need to prove that, for each $\mathfrak{p} \in \Sigma_p$, in the Grothendieck group of $\overline{\mathbb{Q}}_l[\Phi_{\mathfrak{p}^2}]$, we have an equality

$$(7.5.1)\quad \big[(\pi_\mathfrak{p})^{\mathrm{Iw}_\mathfrak{p}}\big] = \sum_{i_\mathfrak{p}=0}^{d_\mathfrak{p}} \sum_{j_\mathfrak{p}=0}^{i_\mathfrak{p}} (-1)^{j_\mathfrak{p}} \Big[\bigoplus_{\substack{\#\mathtt{T}_{/\mathfrak{p}}=i_\mathfrak{p},\\ \text{s.t.}\#I_{\mathtt{T},\mathfrak{p}}=j_\mathfrak{p}}} e_{\mathtt{T}_{/\mathfrak{p}}} \tilde{\rho}_{\pi,\mathfrak{p},l}^{\mathtt{S}(\mathtt{T})_{/\mathfrak{p}}} \otimes (\pi_{\mathtt{S}(\mathtt{T}),\mathfrak{p}})^{K_{\mathtt{S}(\mathtt{T}),\mathfrak{p}}} \otimes \bigotimes_{\tau \in I_{\mathtt{T},\mathfrak{p}}} (\overline{\mathbb{Q}}_l \oplus \overline{\mathbb{Q}}_l f_\tau)\Big],$$

where $\Phi_{\mathfrak{p}^2}$ acts on $(\pi_\mathfrak{p})^{\mathrm{Iw}_\mathfrak{p}}$ by $N_{F/\mathbb{Q}}(\mathfrak{p})^2 S_\mathfrak{p}/U_\mathfrak{p}^2$.

When $\pi_\mathfrak{p}$ is ramified, the left hand side (7.5.1) is nonzero if and only if $\pi_\mathfrak{p}$ is Steinberg, in which case it is one-dimensional with trivial $\Phi_{\mathfrak{p}^2}$-action. All terms on the right hand side of (7.5.1) is zero except when $\mathtt{T}_{/\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}}$ and hence $I_{\mathtt{T},\mathfrak{p}} = \mathtt{S}(\mathtt{T})_{\infty/\mathfrak{p}} - \mathtt{T}_{/\mathfrak{p}} = \emptyset$, in which case the contribution is one-dimensional with trivial $\Phi_{\mathfrak{p}^2}$ if $\pi_\mathfrak{p}$ is Steinberg and zero otherwise. Both sides agree.

It is then left to prove (7.5.1) when $\pi_\mathfrak{p}$ is unramified. Let α and β be two eigenvalues of $\mathrm{Frob}_\mathfrak{p}$ action on $\rho_{\pi,l}$ and let $\{v, w\}$ be a (generalized) eigenbasis corresponding to the two eigenvalues respectively; thus the $S_\mathfrak{p}$-eigenvalue on $\pi_\mathfrak{p}$ is $\omega_\pi(\varpi_\mathfrak{p}^{-1}) = \alpha\beta/p^{d_\mathfrak{p}}$. We take a square root λ of $\alpha\beta/p^{d_\mathfrak{p}}$ and put $\alpha_0 = \alpha/\lambda$ and $\beta_0 = \beta/\lambda$ so that $\alpha_0\beta_0 = p^{d_\mathfrak{p}}$. Then $\Phi_{\mathfrak{p}^2}$ acts on $(\pi_\mathfrak{p})^{\mathrm{Iw}_p}$ with eigenvalues α_0^2 and β_0^2. We need to match this with the right hand side of (7.5.1).

We write $d = d_\mathfrak{p}$, $i = i_\mathfrak{p}$ and $j = j_\mathfrak{p}$ and $\Phi = \Phi_\mathfrak{p}$ to simplify the notation. We label $\Sigma_{\infty/\mathfrak{p}}$ by $\{\tau_1, \dots, \tau_d\}$ so that $\sigma(\tau_i) = \tau_{i+1}$ and $\tau_i = \tau_{i \pmod d}$. We now try to rewrite the right hand of (7.5.1) so that it is easier to work with. We write $\tilde\rho^{\mathtt{S(T)}}_{\pi,\mathfrak{p},l}$ as the vector space $\otimes_{\tau\in\Sigma_{\infty/\mathfrak{p}}-\mathtt{S(T)}_{\infty/\mathfrak{p}}}(\overline{\mathbb{Q}}_l v_\tau\oplus\overline{\mathbb{Q}}_l w_\tau)$, with the convention that the operator Φ acts on symbols by

$$\Phi(v_{\tau_i})=\begin{cases} v_{\tau_{i-1}} & \text{if } i\neq 1,\\ \alpha_0 v_{\tau_n} & \text{if } i=1,\end{cases}\qquad\text{and}\quad \Phi(w_{\tau_i})=\begin{cases} w_{\tau_{i-1}} & \text{if } i\neq 1,\\ \beta_0 w_{\tau_n} & \text{if } i=1.\end{cases}$$

It is straightforward to check that the action of $\Phi^d_{\mathfrak{p}^2}$ on $\tilde\rho^{\mathtt{S(T)}}_{\pi,\mathfrak{p},l}$ is exactly the action of Φ^{2d} on $\otimes_{\tau\in\Sigma_{\infty/\mathfrak{p}}-\mathtt{S(T)}_{\infty/\mathfrak{p}}}(\overline{\mathbb{Q}}_l v_\tau\oplus\overline{\mathbb{Q}}_l w_\tau)$ given as above. We also view these v_τ's and w_τ's as Čech symbols of degree 0; namely, it commutes with other Čech symbols.

We can rewrite the right hand side of (7.5.1) as

(7.5.2)
$$\sum_{i=0}^{d}\sum_{j=0}^{i}(-1)^j\Big[\bigoplus_{\substack{\#\mathtt{T}_{/\mathfrak{p}}=i,\\ \text{s.t.}\#I_{\mathtt{T},\mathfrak{p}}=j}} e_{\mathtt{T}}\bigotimes_{\tau\in\Sigma_{\infty/\mathfrak{p}}-\mathtt{S(T)}_{\infty/\mathfrak{p}}}(\overline{\mathbb{Q}}_l v_\tau\oplus\overline{\mathbb{Q}}_l w_\tau)\otimes(\pi_{\mathtt{S(T)},\mathfrak{p}})^{K_{\mathtt{S(T)},\mathfrak{p}}}\otimes\bigotimes_{\tau\in I_{\mathtt{T},\mathfrak{p}}}(\overline{\mathbb{Q}}_l\oplus\overline{\mathbb{Q}}_l f_\tau)\Big],$$

where Φ sends e_τ to $pe_{\sigma^{-1}\tau}$ and f_τ to $pf_{\sigma^{-1}\tau}$, and it acts trivially on $(\pi_{\mathtt{S(T)},\mathfrak{p}})^{K_{\mathtt{S(T)},\mathfrak{p}}}$. We need to show that (7.5.2), in the Grothendieck group of $\overline{\mathbb{Q}}_l[\Phi]$, is equal to the two dimensional $\overline{\mathbb{Q}}_l$-vector space where Φ acts with eigenvalues α_0 and β_0.

We quickly point out that in each term appearing in the expression above is of the form $\overline{\mathbb{Q}}_l b_{\tau_1}\wedge\cdots\wedge b_{\tau_n}$, where each b_{τ_i} is exactly one of e_{τ_i}, f_{τ_i}, v_{τ_i}, w_{τ_i} or empty.

7.6. Cyclic words. — We now introduce some combinatorial way to describe terms and their contribution in (7.5.2). For each such expression above, we associated a *cyclic word* (of length d), that is a word $(a_1\cdots a_d)$ composed with letters of the following kinds:

- single letters α and β,
- or short combinations: $\alpha'\beta'$, $\overline{\alpha\beta}$, and $\overline{\alpha'\beta'}$,

with the understanding that the last letter is considered adjacent to the first one, and the convention that $a_r=a_{r\ (\mathrm{mod}\ d)}$. The short combinations are viewed as two letters which always come together. For example, $(\bar\beta\alpha\alpha\beta\bar\alpha)$ is a cyclic word, but $(\bar\alpha\alpha\beta\overline{\alpha\beta})$ is not. We may *rotate* each cyclic word by changing it from $(a_1\cdots a_d)$ to $(a_2\cdots a_d a_1)$. The *period* of a cyclic word w, denoted by $\mathrm{per}(w)$, is the minimal $r\in\mathbb{N}$ such that r times rotation of w gives w back. In this case, $(a_1\dots a_r)$ can be also viewed as a cyclic word. We always have $\mathrm{per}(w)|d$. Two words are called *equivalent* if one may be turned into another using rotations. For a cyclic word w, we use $[w]$ to denote its equivalence class.

Now to each term $\overline{\mathbb{Q}}_l b_{\tau_1}\wedge\cdots\wedge b_{\tau_n}$ of (7.5.2), we associate a cyclic word as follows:

- if $b_{\tau_i}=v_{\tau_i}$, we put $a_i=\alpha$;
- if $b_{\tau_i}=w_{\tau_i}$, we put $a_i=\beta$;
- if $b_{\tau_i}=\emptyset$, then $b_{\tau_{i+1}}=e_{\tau_{i+1}}$ (i.e. $\tau_{i+1}\in\mathtt{T}$) and we put $a_ia_{i+1}=\overline{\alpha\beta}$;
- if $b_{\tau_i}=f_{\tau_i}$, then $b_{\tau_{i+1}}=e_{\tau_{i+1}}$ (i.e. $\tau_{i+1}\in\mathtt{T}$) and we put $a_ia_{i+1}=\overline{\alpha'\beta'}$;
- by the description of GO-strata, the only unassigned a_i's (for which we must have $b_{\tau_i}=e_{\tau_i}$) can be partitioned into disjoint union of pairs a_ja_{j+1}, to which we assign $\alpha'\beta'$. (When d is even and $\mathtt{T}_{/\mathfrak{p}}=\Sigma_{\infty/\mathfrak{p}}$, we have exactly two such partitions; we assign two cyclic words in this case. In contrast, when d is odd and $\mathtt{T}_{/\mathfrak{p}}=\Sigma_{\infty/\mathfrak{p}}$, the term in (7.5.2) is trivial because $(\pi_{\mathtt{S(T)},\mathfrak{p}})^{K_{\mathtt{S(T)},\mathfrak{p}}}$-term is zero; this agrees with the fact that there is no cyclic words just consisting of short expressions $\alpha'\beta'$ since d is odd.)

It is somewhat tedious but straightforward to check that this establishes a one-to-one correspondence between terms in (7.5.2) with all cyclic words of length d discussed above.

We now discuss the contribution of the corresponding terms in (7.5.2) to the Grothendieck group of $\overline{\mathbb{Q}}_l[\Phi]$. For a cyclic word w with period $r=\mathrm{per}(w)$, the contribution of the terms of (7.5.2) corresponding to all elements of $[w]$ is given by an r-dimensional representation

(7.6.1)
$$R_{[w]}=(-1)^a[\overline{\mathbb{Q}}_l[\Phi]/(\Phi^r-\lambda)],$$

where a is the number of short combinations $\overline{\alpha\beta}$ and $\overline{\alpha'\beta'}$ in the cyclic word $(a_1\cdots a_d)$, and the number λ is the product of

- α_0 for each α in the cyclic word $(a_1\cdots a_r)$,
- β_0 for each β in the cyclic word $(a_1\cdots a_r)$,
- $\alpha_0\beta_0 = p^d$ for each $\overline{\alpha\beta}$ in the cyclic word $(a_1\cdots a_r)$,
- p^{2d} for each $\alpha'\beta'$ and each $\overline{\alpha'\beta'}$ in the cyclic word $(a_1\cdots a_r)$, and
- a sign, which is nontrivial if and only if d/r is even and there are odd number of pairs of $\overline{\alpha\beta}$ and $\overline{\alpha'\beta'}$ in $(a_1\cdots a_r)$.

This does not depend on the choice of the representative w in the equivalent class $[w]$. For example, the representation associated to the equivalence class of $(\alpha\alpha'\beta'\overline{\alpha\beta}\alpha\alpha'\beta'\overline{\alpha\beta})$ is $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^5+p^{30}\alpha_0)$.

We now need to prove that the total contribution of all cyclic words to (7.5.2) is simply $R_{[\alpha\cdots\alpha]}+R_{[\beta\cdots\beta]}$, which agrees with the contribution from $[(\pi_\mathfrak{p})^{\mathrm{Iw}_\mathfrak{p}}]$. In other words, we need to show that the contribution of all cyclic words except $(\alpha\cdots\alpha)$ and $(\beta\cdots\beta)$ cancel with each other. For this, we need to properly group cyclic words together. We introduce some new terminology.

- For a cyclic word w, its *primitive form*, denoted by $\mathrm{prim}(w)$ is the cyclic word obtained by replacing all $\overline{\alpha\beta}$ by $\alpha\beta$ and all $\overline{\alpha'\beta'}$ by $\alpha'\beta'$. Equivalent cyclic words have equivalent primitive forms. We note that a cyclic word always has (nonstrictly) a bigger period than its primitive form, i.e. $\mathrm{per}(w)\geq\mathrm{per}(\mathrm{prim}(w))$. The upshot of this terminology is that adding overline to either $\alpha\beta$ or $\alpha'\beta'$ will not change the absolute value on λ in (7.6.1) but it will change the sign of $R_{[w]}$.
- We think of the difference between $\alpha\beta$ and $\overline{\alpha\beta}$ as being "conjugate of each other". The same applies to $\alpha'\beta'$ and $\overline{\alpha'\beta'}$. Hence we introduce the convention that $\overline{\overline{\alpha\beta}}=\alpha\beta$ and $\overline{\overline{\alpha\beta}}=\alpha\beta$. The key observation is that, taking the conjugation of a short combination $\alpha\beta$ or $\alpha'\beta'$ will not change the absolute value of λ in (7.6.1) but it will change the sign of $R_{[w]}$. This allows us to cancel the contribution to (7.5.2).

Claim: We group all cyclic words into packages with the same primitive forms up to equivalence. For any equivalent class of primitive forms $[w_0]$ with period $\mathrm{per}(w_0)=r\neq 1$, all cyclic words w with $[\mathrm{prim}(w)]=[w_0]$ have zero total contribution to the sum (7.5.2).

Since the only cyclic words with period 1 are $(\alpha\cdots\alpha)$ and $(\beta\cdots\beta)$, this claim would exactly prove our Theorem 7.1 in the cuspidal case.

Since the Claim can be easily checked when $d=1,2$, We assume $d\geq 3$ from now on. Before proving the claim, we first indicate some simple cases to give the reader some feeling of the argument.

When $r=d$, the claim can be easily deduced as follows. Let w_0 be as in the claim. In this case, every cyclic word w such that $\mathrm{prim}(w)=w_0$ will also have period equal to d. Moreover, w_0 must have at least one adjacent $\alpha\beta$ or a short combination $\alpha'\beta'$. We fix one such, say at ith and $(i+1)$st places. Among all cyclic words whose primitive form is w_0, we may pair those which are identical at all places except at the ith and the $(i+1)$st, where they are conjugate of each other. Their equivalent classes contribute the same representation to the sum (7.5.2), but with different signs (since the signs are determined by the number of pairs of $\overline{\alpha\beta}$ and $\overline{\alpha'\beta'}$). So the total sum would be trivial.

A variant argument of this also works if d/r is odd, as follows. Let w_0 be as in the claim. In this case, if a cyclic word w satisfies $\mathrm{prim}(w)=w_0$, then we only know $r|\mathrm{per}(w)$. But this will not concern us. We fix an adjacent $\alpha\beta$ or $\alpha'\beta'$ in w_0, say at ith and $(i+1)$st places. Without loss of generality, we assume that $i\in\{1,\ldots,r\}$. Then we have adjacent $\alpha\beta$ or $\alpha'\beta'$ at $(sr+i)$th and $(sr+i+1)$st places of w_0 for any $s\in\mathbb{Z}$. For a cyclic word $w=(a_1\cdots a_r)$ whose primitive form is w_0, we define its dual $w^\vee$ to be the following cyclic word

$$w^\vee=(a_1\cdots\overline{a_ia_{i+1}}\cdots\overline{a_{i+r}a_{i+r+1}}\cdots\cdots\overline{a_{i+d-r}a_{i+d-r+1}}\cdots a_d).$$

In other words, we take the conjugation of w at $(sr+i)$th and $(sr+i+1)$st places for all $s \in \mathbb{Z}$. Note that the period of w is still the same as $w^\vee$. Hence their cyclic equivalent classes have the same contribution to the right hand side of (7.5.2), but with signs differed by $(-1)^{d/r} = -1$. So the total contribution is trivial again.

Clearly, a direct generalization of this argument would not work if d/r is even. We look at an example first. Let $w_0 = (\alpha\beta\alpha\beta\alpha\beta\alpha\beta)$. Then the list of cyclic words w with $[\mathrm{prim}(w)] = [w_0]$ and the contribution to (7.5.2) of their equivalence class is given as follows:

(i) equivalence class of $(\alpha\beta\alpha\beta\alpha\beta\alpha\beta)$ contributes $\big[\overline{\mathbb{Q}}_l[\Phi]/(\Phi^2-q)\big]$,
(ii) equivalence class of $(\overline{\alpha\beta}\alpha\beta\alpha\beta\alpha\beta)$ contributes $-\big[\overline{\mathbb{Q}}_l[\Phi]/(\Phi^8-q^4)\big]$,
(iii) equivalence class of $(\overline{\alpha\beta\alpha\beta}\alpha\beta\alpha\beta)$ contributes $\big[\overline{\mathbb{Q}}_l[\Phi]/(\Phi^8-q^4)\big]$,
(iv) equivalence class of $(\overline{\alpha\beta}\alpha\beta\overline{\alpha\beta}\alpha\beta)$ contributes $\big[\overline{\mathbb{Q}}_l[\Phi]/(\Phi^4+q^2)\big]$,
(v) equivalence class of $(\overline{\alpha\beta\alpha\beta\alpha\beta}\alpha\beta)$ contributes $-\big[\overline{\mathbb{Q}}_l[\Phi]/(\Phi^8-q^4)\big]$, and
(vi) equivalence class of $(\overline{\alpha\beta\alpha\beta\alpha\beta\alpha\beta})$ contributes $\big[\overline{\mathbb{Q}}_l[\Phi]/(\Phi^2+q)\big]$,

where $q = p^8$. Note that, in (iii) and (v), the sign on power of p is changed according to the last rule in (7.6.1). One sees that the factorization $\Phi^8 - q^4 = (\Phi^4+q^2)(\Phi^2+q)(\Phi^2-q)$ is used to prove that the total contribution to the sum (7.5.2) is zero.

We now handle the most general case of the claim. We fix an adjacent $\alpha\beta$ or a short combination $\alpha'\beta'$ in w_0, at ith and $(i+1)$st places. Without loss of generality, we assume that $i \in \{1, \dots, r\}$. Assume that $d/r = 2^t s$ for $t \in \mathbb{N}$ and s odd. We fix a positive divisor s'' of s, and write $s' = s/s''$. Let $\mathrm{CW}_{s''}(w_0)$ denote the subset of the cyclic words w of length d whose primitive form is w_0 and its period is of the form $\mathrm{per}(w) = 2^{t''}s''r$ for some integer $0 \le t'' \le t$. For each $j \in \{0, \dots, 2^t-1\}$, we define an operator $\mathtt{r}_j$ on $\mathrm{CW}_{s''}(w_0)$ as follows: for $w = (a_1 \cdots a_d) \in \mathrm{CW}_{s''}(w_0)$, $\mathtt{r}_j(w) = (b_1 \cdots b_d)$ is given by $b_{(m2^t+j)s''r+i}b_{(m2^t+j)s''r+i+1} = \overline{a_{(m2^t+j)s''r+i}a_{(m2^t+j)s''r+i+1}}$ for $0 \le m \le s'-1$, and $b_n = a_n$ for any other n's. It is easy to see that $\mathtt{r}_j(w) \in \mathrm{CW}_{s''}(w_0)$, and two elements of $\mathrm{CW}_{s''}(w_0)$ are equivalent if and only if there images under $\mathtt{r}_j$ are equivalent. Note that $\mathtt{r}_j$ has order 2. Let $\mathfrak{G}_{2^t} \simeq (\mathbb{Z}/2\mathbb{Z})^{2^t}$ denote the group generated by $\mathtt{r}_j$ for $0 \le j \le 2^t-1$. We now group the cyclic words in $\mathrm{CW}_{s''}(w_0)$ into $\mathfrak{G}_{2^t}$-orbits, and as well as their equivalence classes. We have the following

Subclaim: Let $\mathcal{W} \subseteq \mathrm{CW}_{s''}(w_0)$ be a $\mathfrak{G}_{2^t}$-orbit, and let $[\mathcal{W}]$ be the associated set of equivalence classes. Then the contribution of $[\mathcal{W}]$ to the sum (7.5.2) is zero.

It is clear the claim would follow this Subclaim. From now on, we fix such a $\mathcal{W}$. Among all periods of cyclic words in $\mathcal{W}$, there is a minimal one, which we denote by $\tilde{r} = 2^{t''}s''r$. Put $t' = t - t''$ so that $d = (2^{t'}s')\tilde{r}$. For any $w \in \mathcal{W}$, $\mathrm{per}(w)/\tilde{r}$ is a power of 2. We fix an cyclic word $w^\star \in \mathcal{W}$ with period $\tilde{r}$.

The case when $t' = 0$ is simple, which we handle first. This is the case when $d/\tilde{r}$ is odd. Hence the periods for all cyclic words $w \in \mathcal{W}$ are in fact the same. We consider the action of $\mathtt{r}_0$, for example. The $\overline{\mathbb{Q}}_l[\Phi]$-module associated to the equivalence class of w is the same as that of $\mathtt{r}_0(w)$. But their contributions in (7.5.2) differ by a sign since $\mathtt{r}_0$ takes conjugation at s' places. The claim is proved in this case.

From now on, we assume that $t' > 0$. We will have to do an explicit computation. For $k = 0, \dots, t'$, we consider the following set of operators

$$\mathrm{Op}_k = \big\{\mathtt{r}_j \circ \mathtt{r}_{j+2^{t''+k}} \circ \cdots \circ \mathtt{r}_{j+2^t-2^{t''+k}} \;\big|\; j = 0, \dots, 2^{t''+k}-1\big\},$$

and let $\langle \mathrm{Op}_k \rangle \subseteq \mathfrak{G}_{2^t}$ denote the subgroup generated by Op_k. We have $\langle \mathrm{Op}_{k-1} \rangle \subseteq \langle \mathrm{Op}_k \rangle$ and $\langle \mathrm{Op}_{t'} \rangle = \mathfrak{G}_{2^t}$. Let Ob_k denote the orbit of $w^\star$ under the action of $\langle \mathrm{Op}_k \rangle$. It is not hard to see that, for each k, Ob_k consist of exactly those $w \in \mathcal{W}$ such that $\mathrm{per}(w) | 2^{k+t''}s''r$. Put $\mathrm{Ob}_0^\circ = \mathrm{Ob}_0$ and $\mathrm{Ob}_k^\circ = \mathrm{Ob}_k - \mathrm{Ob}_{k-1}$ for $k \ge 1$, and let $[\mathrm{Ob}_k^\circ]$ denote the associated set of equivalence classes. Thus Ob_k° consists of exactly those cyclic words $w \in \mathcal{W}$ such that $\mathrm{per}(w) = 2^{t''+k}s''r$, and we have $[\mathcal{W}] = \bigcup_{k=0}^{t'} [\mathrm{Ob}_k^\circ]$. It is clear that the cardinalities of these sets are

$$\#\mathrm{Ob}_0 = 2^{2^{t''}}, \quad \#\mathrm{Ob}_k^\circ = 2^{2^{t''+k}} - 2^{2^{t''+k-1}} \text{ for } k = 1, \dots, t'.$$

We first look at the contribution from $[\mathrm{Ob}_0]$ to the sum (7.5.2). The equivalent class $w^\star$ corresponds to a representation $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{\tilde{r}} - \lambda)$ for some $\lambda \in \overline{\mathbb{Q}}_l$. Then among all the cyclic words in Ob_0, half of them (i.e. those obtained by applying even number of operators in Op_0 to $w^\star$) correspond to the same representation as $w^\star$, while the other half correspond to $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{\tilde{r}} + \lambda)$. The change of the sign is a result of the last rule in (7.6.1). We note moreover that any two cyclic words in Ob_0 are not equivalent unless they are equal. Hence, the total contribution of $[\mathrm{Ob}_0]$ to (7.5.2) is $2^{2^{t''}-1}$ copies of $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{\tilde{r}} - \lambda)$ and $2^{2^{t''}-1}$ copies of $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{\tilde{r}} + \lambda)$.

Similarly, among the elements of Ob_k° for $k = 1, \dots, t'-1$, $2^{2^{t''+k}-1}$ of them are obtained by applying odd number of operators in Op_k to $w^\star$, while $2^{2^{t''+k}-1} - 2^{2^{t''+k-1}}$ of them are obtained by using even number of operators. By the rules in (7.6.1), the representation corresponding to the elements in the first case is $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{2^k\tilde{r}} + \lambda^{2^k})$, and that for the second case is $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{2^k\tilde{r}} - \lambda^{2^k})$. Since every 2^k elements in Ob_k° give rise to one equivalence class in $[\mathrm{Ob}_k^\circ]$, we see that the multiplicities of the two representations above given by $[\mathrm{Ob}_k^\circ]$ are $2^{2^{t''+k}-1}/2^k = 2^{2^{t''+k}-k-1}$ and $2^{2^{t''+k}-k-1} - 2^{2^{t''+k-1}-k}$ respectively.

For $k = t'$, the representation associated to a cyclic equivalent class of a cyclic word $w^\star$ from $\mathrm{Ob}_{t'}^\circ$ is always $\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{2^{t'}\tilde{r}} - \lambda^{2^{t'}})$ (note that the index of the period is an odd number s'' now). But the contribution to the sum (7.5.2) is the same as the contribution of $w^\star$ if and only if this element is obtained from $w^\star$ by applying even number of operators from $\mathrm{Op}_{t'}$.

In summary, we list all the representations we see in the following table.

k	Representation	Multiplicity	Sign of the Contribution
0	$\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{\tilde{r}} - \lambda)$	$2^{2^{t''}-1}$	same as $w^\star$
0	$\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{\tilde{r}} + \lambda)$	$2^{2^{t''}-1}$	same as $w^\star$
1	$\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{2\tilde{r}} - \lambda^2)$	$2^{2^{t''+1}-1-1} - 2^{2^{t''}-1}$	same as $w^\star$
1	$\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{2\tilde{r}} + \lambda^2)$	$2^{2^{t''+1}-1-1}$	same as $w^\star$
	$\cdots$	$\cdots$	
t'	$\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{2^{t'}\tilde{r}} - \lambda^{2^{t'}})$	$2^{2^t-t'-1} - 2^{2^{t-1}-t'-1}$	same as $w^\star$
t'	$\overline{\mathbb{Q}}_l[\Phi]/(\Phi^{2^{t'}\tilde{r}} - \lambda^{2^{t'}})$	$2^{2^t-t'-1}$	opposite to $w^\star$

Then using the factorization

$$\Phi^{2^{t'}\tilde{r}} - \lambda^{2^{t'}} = (\Phi^{2^{t'-1}\tilde{r}} + \lambda^{2^{r'-1}}) \cdots (\Phi^{\tilde{r}} + \lambda)(\Phi^{\tilde{r}} - \lambda),$$

it follows immediately that the total contribution of all $[\mathrm{Ob}_k^\circ]$ for $0 \le k \le t'$ to (7.5.2) is zero. This finishes the proof of the Subclaim, and hence also Theorem 7.1.

Remark 7.7. — Our proof of Theorem 7.1 in the cuspidal case is rather combinatorial, basically by brutal force. It would be great if one can give a more conceptual or geometric proof.

If one needs only the action of high power of $\Phi_{\mathfrak{p}^2}$, the proof can be significantly simplified. In fact, this suffices for the application to proving classicality result as stated in Corollary 7.2. Nonetheless, we feel that Theorem 7.1 has its own importance; it deserves a trying for the most optimal statement.

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