AN UPPER BOUND FOR THE ABBES-SAITO FILTRATION OF FINITE FLAT GROUP SCHEMES AND APPLICATIONS

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ABSTRACT. Let $\mathcal{O}_K$ be a complete discrete valuation ring of residue characteristic $p > 0$, and $G$ be a finite flat group scheme over $\mathcal{O}_K$ of order a power of $p$. We prove in this paper that the Abbes-Saito filtration of $G$ is bounded by a linear function of the degree of $G$. Assume $\mathcal{O}_K$ has generic characteristic 0 and the residue field of $\mathcal{O}_K$ is perfect. Fargues constructed the higher level canonical subgroups for a “near from being ordinary” Barsotti-Tate group $\mathcal{G}$ over $\mathcal{O}_K$. As an application of our bound, we prove that the canonical subgroup of $\mathcal{G}$ of level $n \geq 2$ constructed by Fargues appears in the Abbes-Saito filtration of the $p^n$-torsion subgroup of $\mathcal{G}$.

Let $\mathcal{O}_K$ be a complete discrete valuation ring with residue field $k$ of characteristic $p > 0$ and fraction field $K$. We denote by $\nu_\pi$ the valuation on $K$ normalized by $\nu_\pi(K^\times) = \mathbb{Z}$. Let $G$ be a finite and flat group scheme over $\mathcal{O}_K$ of order a power of $p$ such that $G \otimes K$ is étale. We denote by $(G^a, a \in \mathbb{Q}_{\geq 0})$ the Abbes-Saito filtration of $G$. This is a decreasing and separated filtration of $G$ by finite and flat closed subgroup schemes. We refer the readers to [AS02, AS03, AM04] for a full discussion, and to section 1 for a brief review of this filtration. Let $\omega_G$ be the module of invariant differentials of $G$. The generic étaleness of $G$ implies that $\omega_G$ is a torsion $\mathcal{O}_K$-module of finite type. There exist thus nonzero elements $a_1, \cdots, a_d \in \mathcal{O}_K$ such that

$$\omega_G \simeq \bigoplus_{i=1}^d \mathcal{O}_K/(a_i).$$

We put $\deg(G) = \sum_{i=1}^d \nu_\pi(a_i)$, and call it the degree of $G$. The aim of this note is to prove the following

**Theorem 0.1.** Let $G$ be a finite and flat group scheme over $\mathcal{O}_K$ of order a power of $p$ such that $G \otimes K$ is étale. Then we have $G^a = 0$ for $a > \frac{p}{p-1} \deg(G)$.

Our bound is quite optimal when $G$ is killed by $p$. Let $E_\delta = \text{Spec}(\mathcal{O}_K[X]/(X^p - \delta X))$ be the group scheme of Tate-Oort over $\mathcal{O}_K$. We have $\deg(E_\delta) = \nu_\pi(\delta)$, and an easy computation by Newton polygons gives [Fa09, Lemme 5]

$$E_\delta^a = \begin{cases} E_\delta & \text{if } 0 \leq a \leq \frac{p}{p-1} \deg(E_\delta) \\ 0 & \text{if } a > \frac{p}{p-1} \deg(E_\delta). \end{cases}$$
However, our bound may be improved when $G$ is not killed by $p$ or $G$ contains many identical copies of a closed subgroup. In [Hat06, Thm. 7], Hattori proves that if $K$ has characteristic 0 and $G$ is killed by $p^n$, then the Abbes-Saito filtration of $G$ is bounded by that of the multiplicative group $\mu_p$, i.e., we have $G^a = 0$ if $a > \frac{e}{p^e} - 1$ where $e$ is the absolute ramification index of $K$. Compared with Hattori's result, our bound has the advantage that it works in both characteristic 0 and characteristic $p$, and that it is good if $\deg(G)$ is small.

The basic idea to prove 0.1 is to approximate general power series over $O_K$ by linear functions. First, we choose a “good” presentation of the algebra of $G$ such that the defining equations of $G$ involve only terms of total degree $m(p-1) + 1$ with $m \in \mathbb{Z}_{\geq 0}$ (Prop. 1.6). The existence of such a presentation is a consequence of the classical theory on $p$-typical curves of formal groups. With this good presentation, we can prove that the neutral connected component of the $a$-tubular neighborhood of $G$ is isomorphic to a closed rigid ball for $a > \frac{p^e}{p^e-1} \deg(G)$ (Lemma 1.9), and the only zero of the defining equations of $G$ in the neutral component is the unit section.

The motivation of our theorem comes from the theory of canonical subgroups. We assume that $K$ has characteristic 0, and the residue field $k$ is perfect of characteristic $p \geq 3$. Let $G$ be a Barsotti-Tate group of dimension $d \geq 1$ over $O_K$. If $G$ comes from an abelian scheme over $O_K$, the canonical subgroup of level 1 of $G$ was first constructed by Abbes and Mokrane in [AM04]. Then the author generalized their result to the Barsotti-Tate case [Ti06]. We actually proved that if a Barsotti-Tate group $G$ over $O_K$ is “near from being ordinary”, a condition expressed explicitly as a bound on the Hodge height of $G$ (cf. 2.1), then a certain piece of the Abbes-Saito filtration of $G[p]$ lifts the kernel of Frobenius of the special fiber of $G$ [Ti06, Thm. 1.4]. Later on, Fargues [Fa09] gave another construction of the canonical subgroup of level 1 using Hodge-Tate maps, and his approach also allowed us to construct by induction the canonical subgroups of level $n \geq 2$, i.e., the canonical lifts of the kernel of $n$-th iteration of the Frobenius. He proved that the canonical subgroup of higher level appears in the Harder-Narasihman filtration of $G[p^n]$, which was introduced by him in [Fa07]. It is conjectured that the canonical subgroup of higher level also appears in the Abbes-Saito filtration of $G[p^n]$. In this paper, we prove this conjecture as a corollary of 0.1 (Thm. 2.5). Fargues’s result on the degree of the quotient of $G[p^n]$ by its canonical subgroup of level $n$ (see Thm. 2.4(i)) will play an essential role in our proof.

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0.3. Notation. In this paper, $O_K$ will denote a complete discrete valuation ring with residue field $k$ of characteristic $p > 0$, and with fraction field $K$. Let $\pi$ be a uniformizer of $O_K$, and $v_\pi$ be the valuation on $K$ normalized by $v_\pi(\pi) = 1$. Let $K$ be an algebraic closure of $K$, $K^{sep}$ be the separable closure of $K$ contained in $\overline{K}$, and $G_K$ be the Galois group $\text{Gal}(K^{sep}/K)$. We denote still by $v_\pi$ the unique extension of the valuation to $\overline{K}$.
1. Proof of Theorem 0.1

We recall first the definition of the filtration of Abbes-Saito for finite flat group schemes according to [AM04, AS03].

1.1. For a semi-local ring $R$, we denote by $\mathfrak{m}_R$ its Jacobson radical. An algebra $R$ over $\mathcal{O}_K$ is called formally of finite type, if $R$ is semi-local, complete with respect to the $\mathfrak{m}_R$-adic topology, Noetherian and $R/\mathfrak{m}_R$ is finite over $k$. We say an $\mathcal{O}_K$-algebra $R$ formally of finite type is formally smooth, if each of the factors of $R$ is formally smooth over $\mathcal{O}_K$.

Let $\text{FEA}_{\mathcal{O}_K}$ be the category of finite, flat and generically étale $\mathcal{O}_K$-algebras, and $\text{Set}_{G_K}$ be the category of finite sets endowed with a discrete action of the Galois group $G_K$. We have the fiber functor

$$\mathcal{F} : \text{FEA}_{\mathcal{O}_K} \to \text{Set}_{G_K},$$

which associates with an object $A$ of $\text{FEA}_{\mathcal{O}_K}$ the set $\text{Spec}(A)(\overline{K})$ equipped with the natural action of $G_K$. We define a filtration on the functor $\mathcal{F}$ as follows. For each object $A$ in $\text{FEA}_{\mathcal{O}_K}$, we choose a presentation

$$0 \to I \to \mathcal{A} \to A \to 0,$$

(1.1.1)

where $\mathcal{A}$ is an $\mathcal{O}_K$-algebra formally of finite type and formally smooth. For any $a = \frac{m}{n}$ with $m$ prime to $n$, we define $\mathcal{A}^a$ to be the $\pi$-adic completion of the subring $\mathcal{A}[\pi^n/\pi^m] \subset \mathcal{A} \otimes_{\mathcal{O}_K} K$ generated over $\mathcal{A}$ by all the $f/\pi^n$ with $f \in I^n$. The $\mathcal{O}_K$-algebra $\mathcal{A}^a$ is topologically of finite type, and the tensor product $\mathcal{A}^a \otimes_{\mathcal{O}_K} K$ is an affinoid algebra over $K$ [AS03, Lemma 1.4]. We put $X^a = \text{Sp}(\mathcal{A}^a \otimes_{\mathcal{O}_K} K)$, which is a smooth affinoid variety over $K$ [AS03, Lemma 1.7]. We call it the $a$-th tubular neighborhood of $\text{Spec}(A)$ with respect to the presentation (1.1.1). The $G_K$-set of the geometric connected components of $X^a$, denoted by $\pi_0(X^a(A)_{\overline{K}})$, depends only on the $\mathcal{O}_K$-algebra $A$ and the rational number $a$, but not on the choice of the presentation [AS03, Lemma 1.9.2]. For rational numbers $b > a > 0$, we have natural inclusions of affinoid varieties $\text{Sp}(A \otimes_{\mathcal{O}_K} K) \hookrightarrow X^b \hookrightarrow X^a$, which induce natural morphisms $\text{Spec}(A)(\overline{K}) \to \pi_0(X^b(A)_{\overline{K}}) \to \pi_0(X^a(A)_{\overline{K}})$. For a morphism $A \to B$ in $\text{FEA}_{\mathcal{O}_K}$, we can choose proper presentations of $A$ and $B$ so that we have a functorial map $\pi_0(X^a(B)_{\overline{K}}) \to \pi_0(X^a(A)_{\overline{K}})$. Hence we get, for any $a \in \mathbb{Q}_{>0}$, a (contravariant) functor

$$\mathcal{F}^a : \text{FEA}_{\mathcal{O}_K} \to \text{Set}_{G_K}$$

given by $A \mapsto \pi_0(X^a(A)_{\overline{K}})$. We have natural morphisms of functors $\phi_a : \mathcal{F} \to \mathcal{F}^a$, and $\phi_{a,b} : \mathcal{F}^b \to \mathcal{F}^a$ for rational numbers $b > a > 0$ with $\phi_a = \phi_{b,a} \circ \phi_b$. For any $A$ in $\text{FEA}_{\mathcal{O}_K}$, we have $\mathcal{F}(A) \xrightarrow{\sim} \lim_{\leftarrow a \in \mathbb{Q}_{>0}} \mathcal{F}^a(A)$ [AS02, 6.4]; if $A$ is a complete intersection over $\mathcal{O}_K$, the map $\mathcal{F}(A) \to \mathcal{F}^a(A)$ is surjective for any $a$ [AS02, 6.2].

1.2. Let $G = \text{Spec}(A)$ be a finite and flat group scheme over $\mathcal{O}_K$ such that $G \otimes K$ is étale over $K$, and $a \in \mathbb{Q}_{>0}$. The group structure of $G$ induces a group structure on $\mathcal{F}(A)$, and the natural map $G(\overline{K}) = \mathcal{F}(A) \to \mathcal{F}^a(A)$ is a homomorphism of groups. Hence the kernel $G^a(\overline{K})$ of $G(\overline{K}) \to \mathcal{F}^a(A)$ is a $G_K$-invariant subgroup of $G(\overline{K})$, and it defines a closed subgroup scheme $G^a_K$ of the generic fiber $G \otimes K$. The scheme theoretic closure of $G^a_K$ in $G$, denoted by $G^a$, is a closed subgroup of $G$ finite and flat over $\mathcal{O}_K$. Putting $G^0 = G$,
we get a decreasing and separated filtration \((G^a, a \in \mathbb{Q}_{\geq 0})\) of \(G\) by finite and flat closed subgroup schemes. We call it *Abbes-Saito filtration* of \(G\). For any real number \(a \geq 0\), we put \(G^{a+} = \bigcup_{b \in \mathbb{Q}_{>a}} G^b\).

Assume \(G\) is connected, i.e., the ring \(A\) is local. Let

\[
0 \rightarrow I \rightarrow O_K[[X_1, \cdots, X_d]] \rightarrow A \rightarrow 0
\]

be a presentation of \(A\) by the ring of formal power series such that the unit section of \(G\) corresponds to the point \((X_1, \cdots, X_d) = (0, \cdots, 0)\). Since \(A\) is a relative complete intersection over \(O_K\), \(I\) is generated by \(d\) elements \(f_1, \cdots, f_d\). For \(a \in \mathbb{Q}_{>0}\), the \(K\)-valued points of the \(a\)-th tubular neighborhood of \(G\) are given by

\[
X^a(K) = \{(x_1, \cdots, x_d) \in m_K^d | v_a(f_i(x_1, \cdots, x_d)) \geq a \text{ for } 1 \leq i \leq d\},
\]

where \(m_K\) is the maximal ideal of \(O_K\). The subset \(G(\overline{K}) \subset X^a(\overline{K})\) corresponds to the zeros of the \(f_i\)'s. Let \(X^a_0\) be the connected component of \(X^a\) containing 0. Then the subgroup \(G^a(\overline{K})\) is the intersection of \(X^a_0(\overline{K})\) with \(G(\overline{K})\).

The basic properties of Abbes-Saito filtration that we need are summarized as follows.

**Proposition 1.3** ([AM04] 2.3.2, 2.3.5). Let \(G\) and \(H\) be finite and flat group schemes, generically étale over \(O_K\), \(f : G \rightarrow H\) be a homomorphism of group schemes.

(i) \(G^{0+}\) is the connected component of \(G\), and we have \((G^{0+})^a = G^a\) for any \(a \in \mathbb{Q}_{>0}\).

(ii) For \(a \in \mathbb{Q}_{>0}\), \(f\) induces a canonical homomorphism \(f^a : G^a \rightarrow H^a\). If \(f\) is flat and surjective, then \(f^a(\overline{K}) : G^a(\overline{K}) \rightarrow H^a(\overline{K})\) is surjective.

Now we return to the proof of Theorem 0.1.

**Lemma 1.4.** Let \(R\) be a \(\mathbb{Z}_p\)-algebra, \(\mathcal{X}\) be a formal group of dimension \(d\) over \(R\) such that \(\text{Lie}(\mathcal{X})\) is a free \(R\)-module of rank \(d\). Then

(i) the ring \(\mathbb{Z}_p\) acts naturally on \(\mathcal{X}\), and its image in \(\text{End}_R(\mathcal{X})\) lies in the center of \(\text{End}_R(\mathcal{X})\);

(ii) there exist parameters \((X_1, \cdots, X_d)\) of \(\mathcal{X}\), such that we have \([\zeta](X_1, \cdots, X_d) = (\zeta X_1, \cdots, \zeta X_d)\) for any \((p - 1)\)-th root of unity \(\zeta \in \mathbb{Z}_p\).

*Proof.* This is actually a classical result on formal groups. In the terminology of [Haz78], the formal group \(\mathcal{X}\) comes from the base change of \(\mathcal{X}^{\text{univ}}\) defined by the \(d\)-dimensional universal \(p\)-typical formal group law (denoted by \(F_V(X, Y)\) in [Haz78, 15.2.8]) over \(\mathbb{Z}_p[V] = \mathbb{Z}_p[V_i(j, k); i \in \mathbb{Z}_{>0}, j, k = 1, \cdots, d]\), where the \(V_i(j, k)\)'s are free variables. So we are reduced to proving the Lemma for \(\mathcal{X}^{\text{univ}}\). If \(X\) and \(Y\) are short for the column vectors \((X_1, \cdots, X_d)\) and \((Y_1, \cdots, Y_d)\) respectively, the formal group law on \(\mathcal{X}^{\text{univ}}\) is determined by

\[
F_V(X, Y) = f_V^{-1}(f_V(X) + f_V(Y)), \quad \text{with } f_V(X) = \sum_{i=0}^{\infty} a_i(V) X^p^i,
\]

where \(a_i(V)\)'s are certain \(d \times d\) matrices with coefficients in \(\mathbb{Q}_p[V]\) with \(a_1(V)\) invertible, \(X^{p^i}\) is short for \((X^{p^1}, \cdots, X^{p^d})\), and \(f_V^{-1}\) is the unique \(d\)-tuple of power series in \((X_1, \cdots, X_d)\) with coefficients in \(\mathbb{Q}_p[V]\) such that \(f_V^{-1} \circ f_V = 1\) [Haz78, 10.4]. We note that \(F_V(X, Y)\) is a \(d\)-tuple of power series with coefficient in \(\mathbb{Z}_p[V]\), although \(f_V(X)\) has coefficients in \(\mathbb{Q}_p[V]\).
[Haz78, 10.2(i)]. Via approximation by integers, we see easily that the multiplication by an element $\xi \in \mathbb{Z}_p$ can be well defined as $[\xi](X) = f_{iV}^{-1}(\xi f_{iV}(X))$. This proves (i). Statement (ii) is an immediate consequence of the fact that $f_{iV}(X)$ involves just $p$-powers of $X$. \hfill $\square$

**Remark 1.5.** The referee gives the following alternative proof of this Lemma via the Cartier theory of formal groups. Let $\mathcal{X}$ be the formal group over $R$ as in the Lemma. We denote by $\mathcal{X}(R[[T]])$ the group of $R[[T]]$-valued points of $\mathcal{X}$ whose reduction modulo $T$ is the neutral element $0 \in \mathcal{X}(R)$. A formal group law over $\mathcal{X}$ is a datum $(\mathcal{X}; \gamma_1, \ldots, \gamma_d)$, where $\gamma_1, \ldots, \gamma_d \in \mathcal{X}(R[[T]])$ are such that their image in $\mathcal{X}(R[T]/T^2)$ forms a basis of $\text{Lie}(\mathcal{X})$. In particular, $(\gamma_i)_{1 \leq i \leq d}$ establish an isomorphism of formal schemes over $R$ $\mathcal{X} \cong \text{Spf}(R[[X_1, \ldots, X_d]])$. Recall that $\mathcal{X}(R[[T]])$ is the Cartier module associated with $\mathcal{X}$ over the big Cartier ring (denoted by $\text{Cart}(R)$ in [Ch94, 2.3]). Since $R$ is a $\mathbb{Z}_p$-algebra, the Cartier theory [Ch94, 4.3, 4.4] implies that there exists a $p$-typical formal group law $(\mathcal{X}; \gamma_1, \ldots, \gamma_d)$ over $\mathcal{X}$, i.e. we have $\epsilon_p \cdot \gamma_i = 0$, where

$$
\epsilon_p = \prod_{\ell \text{ prime}} \left( 1 - \frac{1}{\ell} V_{\ell} \ell \right)
$$

is Cartier’s idempotent in $\text{Cart}(R)$ (see [Ch94, 4.1]). Let $\Delta : \mathcal{X}_p = W(F_p) \rightarrow W(\mathbb{Z}_p)$ be the Cartier homomorphism given by $(x_0, x_1, \ldots) \mapsto ([x_0], [x_1], \ldots)$, where $x_n \in F_p$ and $[x_n]$ denotes its Teichmüller lift. Then we get a natural map $u : \mathcal{X}_p \xrightarrow{\Delta} W(\mathbb{Z}_p) \rightarrow W(R)$. For a $(p-1)$-th root of unity $\zeta \in \mathbb{Z}_p$, we have $u(\zeta) = [\zeta] \in W(R)$. Note that for any $a \in R$ and $1 \leq i \leq d$, the $p$-typical curve $[a] \cdot \gamma_i$ is the image of $\gamma_i$ under the map $\mathcal{X}(R[[T]]) \rightarrow \mathcal{X}(R[[T]])$ induced by $T \mapsto aT$. Applying this fact to $u(\zeta) \cdot \gamma_i = [\zeta] \cdot \gamma_i$, one obtains the Lemma immediately.

**Proposition 1.6.** Let $G = \text{Spec}(A)$ be a connected finite and flat group scheme over $\mathcal{O}_K$ of order a power of $p$. Then there exists a presentation of $A$ of type (1.2.1) such that the defining equations $f_i$ for $1 \leq i \leq d$ have the form

$$
f_i(X_1, \ldots, X_d) = \sum_{|\underline{n}| \geq 1} a_{i, \underline{n}} X^\underline{n} \quad \text{with } a_{i, \underline{n}} = 0 \text{ if } (p-1) \not| (|\underline{n}| - 1),
$$

where $\underline{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 0})^d$ are multi-indexes, $|\underline{n}| = \sum_{j=1}^d n_j$, and $X^\underline{n}$ is short for $\prod_{j=1}^d X_j^{n_j}$.

**Proof.** By a theorem of Raynaud [BBM82, 3.1.1], there is a projective abelian variety $V$ over $\mathcal{O}_K$, and an embedding of group schemes $j : G \hookrightarrow V$. Let $V'$ be the quotient of $V$ by $G$. Let $\mathcal{X}, \mathcal{Y}$ be respectively the formal completion of $V$ and $V'$ along their unit sections. They are formal groups over $\mathcal{O}_K$. Since $G$ is connected, it’s identified with the kernel of the natural isogeny $\phi : \mathcal{X} \rightarrow \mathcal{Y}$. Let $(X_1, \ldots, X_d)$ (resp. $(Y_1, \ldots, Y_d)$) be parameters of $\mathcal{X}$ (resp. $\mathcal{Y}$) satisfying the preceding lemma. The isogeny $\phi$ is thus given by

$$(X_1, \ldots, X_d) \mapsto (f_1(X_1, \ldots, X_d), \ldots, f_d(X_1, \ldots, X_d)).$$
where \( f_i = \sum_{|n| \geq 1} a_{i,n} X^n \in \mathcal{O}_K[[X_1, \ldots, X_d]]. \) Since for any \((p - 1)\)-th root of unity \( \zeta \in \mathbb{Z}_p \) we have \( f_i(\zeta X_1, \ldots, \zeta X_d) = \zeta f_i(X_1, \ldots, X_d) \), it's easy to see that \( a_{i,n} = 0 \) if \((p - 1) \mid (|n| - 1)\).

\[ \square \]

**Remark 1.7.** As pointed out by the referee, we can avoid using Raynaud’s deep theorem to realize \( G \) as the kernel of an isogeny of formal groups over \( \mathcal{O}_K \). In fact, by the biduality formula \( G \simeq (G^D)^D \), where \( G^D \) denotes the Cartier dual of \( G \), we have a canonical closed embedding \( u: G \hookrightarrow U = \text{Res}_{G/D/S}(G_m) \) of group schemes over \( S = \text{Spec}(\mathcal{O}_K) \). Here, \( \text{“Res}_{G/D/S} \)’ means Weil’s restriction of scalars, so \( U \) is an affine smooth group scheme over \( S \). Since the quotient of an affine scheme by a finite flat group scheme is always representable by a scheme [Ra67], we can consider the quotient \( U'/U \) and the formal groups \( \mathcal{X}, \mathcal{Y} \) associated with \( U \) and \( U' \), so that \( G \) is the kernel of the natural isogeny \( \phi: \mathcal{X} \to \mathcal{Y} \).

1.8. **Proof of Theorem 0.1.** Let \( H = G^{d+} \) be the connected component of \( G \). By 1.3(i), we have \( G^a = H^a \) for \( a \in \mathbb{Q}_{>0} \). The exact sequence of finite flat group schemes \( 0 \to H \to G \to G/H \to 0 \) induces a long exact sequence of finite \( \mathcal{O}_K \)-modules

\[ 0 \to H^{-1}(\ell_{G/H}) \to H^{-1}(\ell_G) \to H^{-1}(\ell_H) \to \omega_{G/H} \to \omega_G \to \omega_H \to 0, \]

where \( \ell_G \) means the co-Lie complex of \( G \) [BBM82, 3.2.9]. Since the generic fiber of \( G/H \) is étale, it’s easy to see that \( H^{-1}(\ell_H) = 0 \). It follows thus that \( 0 \to \omega_{G/H} \to \omega_G \to \omega_H \to 0 \) is exact. Since \( G/H \) is étale, we have \( \omega_{G/H} = 0 \) and hence \( \deg(G) = \deg(H) \). Up to replacing \( G \) by \( H \), we may assume that \( G = \text{Spec}(A) \) is connected.

We choose a presentation of \( A \) as in Prop. 1.6 so that we have an isomorphism of \( \mathcal{O}_K \)-algebras

\[ A \simeq \mathcal{O}_K[[X_1, \ldots, X_d]]/(f_1, \ldots, f_d) \]

where

\[ f_i(X_1, \ldots, X_d) = \sum_{j=1}^d a_{i,j} X_j + \sum_{|n| \geq p} a_{i,n} X^n. \]

As \( A \) is finite as an \( \mathcal{O}_K \)-module, we have

\[ \Omega^1_{A/\mathcal{O}_K} = \Omega^1_{A/\mathcal{O}_K} \simeq \left( \bigoplus_{i=1}^d A \ dX_i \right)/(df_1, \ldots, df_d). \]

Since \( \omega_G \simeq e^*\left( \Omega^1_{A/\mathcal{O}_K} \right) \), where \( e \) is the unit section of \( G \), we get

\[ \omega_G \simeq \left( \bigoplus_{i=1}^d \mathcal{O}_K dX_i \right)/(\sum_{1 \leq j \leq d} a_{i,j} dX_j)_{1 \leq i \leq d}. \]

In particular, if \( U \) denotes the matrix \((a_{i,j})_{1 \leq i, j \leq d} \), then we have \( \deg(G) = v_\pi(\det(U)) \).

For any rational number \( \lambda \), we denote by \( \mathbb{D}^d(0, |\pi|^\lambda) \) (resp. \( \mathbb{D}^d(0, |\pi|^0) \)) the rigid analytic closed (resp. open) disk of dimension \( d \) over \( K \) consisting of points \((x_1, \ldots, x_d)\) with \( v_\pi(x_i) \geq \lambda \) (resp. \( v_\pi(x_i) > \lambda \)) for \( 1 \leq i \leq d \); we put \( \mathbb{D}^d(0, 1) = \mathbb{D}^d(0, |\pi|^0) \) and \( \mathbb{D}^d(0, 1) = \mathbb{D}^d(0, |\pi|^0) \). Let \( a > \frac{p}{\deg(G)} \) be a rational number, \( X^a \) be the \( a \)-th tubular neighborhood
of $G$ with respect to the chosen presentation. By (1.2.2), we have a cartesian diagram of rigid analytic spaces

$$(1.8.1) \quad X^a \xrightarrow{f} \mathbb{D}^d(0, 1) \xrightarrow{g} \mathbb{D}^d(0, |\pi|^a),$$

where horizontal arrows are inclusions, and $f(y_1, \cdots, y_d) = (f_1(y_1, \cdots, y_d), \cdots, f_d(y_1, \cdots, y_d))$. Let $X^a_0$ be the connected component of $X^a$ containing 0. By the discussion below (1.2.2), we just need to prove that 0 is the only zero of the $f_i$'s contained in $X^a_0$.

Let $V = (b_{i,j})_{1 \leq i,j \leq d}$ be the unique $d \times d$ matrix with coefficients in $\mathcal{O}_K$ such that $U = UV = \det(U)I_d$, where $I_d$ is the $d \times d$ identity matrix. If $A^d_K$ denotes the $d$-dimensional rigid affine space over $K$, then $V$ defines an isomorphism of rigid spaces

$$g : A^d_K \to A^d_K; \quad (x_1, \cdots, x_d) \mapsto (\sum_{j=1}^d b_{1,j} x_j, \cdots, \sum_{j=1}^d b_{d,j} x_j).$$

It's clear that $g(\mathbb{D}^d(0, 1)) \subset \mathbb{D}^d(0, 1)$, so that $f$ is defined on $g(\mathbb{D}^d(0, 1))$. The composite morphism $f \circ g : \mathbb{D}^d(0, 1) \to \mathbb{D}^d(0, 1)$ is given by

$$(1.8.2) \quad (x_1, \cdots, x_d) \mapsto (\det(U) x_1 + R_1, \cdots, \det(U) x_d + R_d),$$

where $R_i = \sum_{|a| \geq p} a_{i,k} \prod_{j=1}^d (\sum_{k=1}^d b_{j,k} x_k)^{|a|}$ involves only terms of order $\geq p$ for $1 \leq i \leq d$. For $1 \leq i \leq d$, we have basic estimations

$$(1.8.3) \quad v_\pi(\det(U) x_i) = \deg(G) + v_\pi(x_i) \quad \text{and} \quad v_\pi(R_i) \geq p \min_{1 \leq j \leq d} \{v_\pi(x_j)\}.$$  

**Lemma 1.9.** For any rational number $a > \frac{p}{p-1} \deg(G)$, the map $g$ induces an isomorphism of affinoid rigid spaces

$$g : D^d(0, |\pi|^{-\deg(G)}) \xrightarrow{\sim} X^a_0.$$  

Assuming this Lemma for a moment, we can complete the proof of 0.1 as follows. Consider the composite

$$h = f \circ g|_{D^d(0, |\pi|^{-\deg(G)})} : D^d(0, |\pi|^{-\deg(G)}) \xrightarrow{\sim} X^a_0 \xrightarrow{f} D^d(0, |\pi|^a).$$

In order to complete the proof of 0.1, we just need to prove that the inverse image $h^{-1}(0) = \{0\}$. Let $(x_1, \cdots, x_d)$ be a point of $D^d(0, |\pi|^{-\deg(G)})$, and $(z_1, \cdots, z_d) = h(x_1, \cdots, x_d)$. We may assume $v_\pi(x_i) = \min_{1 \leq i \leq d} \{v_\pi(x_i)\}$. We have $v_\pi(x) \geq a - \deg(G) > \frac{1}{p-1} \deg(G)$ by the assumption on $a$. It follows thus from (1.8.3) that

$$v_\pi(R_1) \geq pv_\pi(x_1) > \deg(G) + v_\pi(x_1) = v_\pi(\det(U)x_1).$$

Hence, we deduce from (1.8.2) that $v_\pi(z_1) = \deg(G) + v_\pi(x_1)$. In particular, $z_1 = 0$ if and only if $x_1 = 0$. Therefore, we have $h^{-1}(0) = \{0\}$. This achieves the proof of Theorem 0.1.
Proof of 1.9. Let $\epsilon$ be any rational number with $0 < \epsilon < \frac{p-1}{p} a - \deg(G)$. We will prove that
\[ D^d(0, |\pi|^{a-\deg(G)}) = D^d(0, |\pi|^{a-\deg(G)-\epsilon}) \cap g^{-1}(X^a). \]
This will imply that $D^d(0, |\pi|^{a-\deg(G)})$ is a connected component of $g^{-1}(X^a)$. Since $g : A_K^d \to A_K^d$ is an isomorphism, the lemma will follow immediately.

We prove first the inclusion “$\supset$”. It suffices to show $g(D^d(0, |\pi|^{a-\deg(G)}) \subseteq X^a$.

Let $(x_1, \cdots, x_d)$ be a point of $D^d(0, |\pi|^{a-\deg(G)})$. By (1.8.1), we have to check that $(z_1, \cdots, z_d) = f(g(x_1, \cdots, x_d))$ lies in $D^d(0, |\pi|^a)$. We get from (1.8.3) that $v_\pi(\det(U)x_i) = \deg(G) + v_\pi(x_i) \geq a$ and $v_\pi(R_i) \geq p(a - \deg(G))$. As $a > \frac{p-1}{p} \deg(G)$, we have $v_\pi(R_i) > a$. It follows from (1.8.2) that
\[ v_\pi(z_i) \geq \min\{v_\pi(\det(U)x_i), v_\pi(R_i)\} \geq a. \]
This proves $(z_1, \cdots, z_d)$ is contained in $D^d(0, |\pi|^a)$, hence we have $g(D^d(0, |\pi|^{a-\deg(G)})) \subseteq X^a$.

To prove the inclusion “$\supset$”, we just need to verify that every point in $D^d(0, |\pi|^{a-\deg(G)-\epsilon})$ but outside $D^d(0, |\pi|^{a-\deg(G)})$ does not lie in $g^{-1}(X^a)$. Let $(x_1, \cdots, x_d)$ be such a point. We may assume that
\[(1.9.1) \ a - \deg(G) - \epsilon \leq v_\pi(x_i) < a - \deg(G) \quad \text{and} \quad v_\pi(x_i) \geq a - \deg(G) - \epsilon \quad \text{for} \ 2 \leq i \leq d.
\]
Let $(z_1, \cdots, z_d) = (\det(U)x_1 + R_d, \cdots, \det(U)x_d + R_d)$ be the image of $(x_1, \cdots, x_d)$ under the composite $f \circ g$. According to (1.8.1), the proof will be completed if we can prove that $(z_1, \cdots, z_d)$ is not in $D^d(0, |\pi|^a)$. From (1.8.3) and (1.9.1), we get $v_\pi(\det(U)x_1) = \deg(G) + v_\pi(x_1) < a$ and $v_\pi(R_i) \geq p(a - \deg(G) - \epsilon)$. Thanks to the assumption on $\epsilon$, we have $p(a - \deg(G) - \epsilon) > a$, so $v_\pi(z_1) = v_\pi(\det(U)x_1) < a$. This shows that $(z_1, \cdots, z_d)$ is not in $g^{-1}(X^a)$, hence the proof of the lemma is complete.

\[ \square \]

2. Applications to Canonical subgroups

In this section, we suppose the fraction field $K$ has characteristic 0 and the residue field $k$ is perfect of characteristic $p \geq 3$. Let $e$ be the absolute ramification index of $O_K$. For any rational number $\epsilon > 0$, we denote by $O_{K,\epsilon}$ the quotient of $O_K$ by the ideal consisting of elements with $p$-adic valuation greater or equal to $\epsilon$.

2.1. First we recall some results on the canonical subgroups according to [AM04], [Ti06] and [Fa09]. Let $v_p : O_K/p \to [0, 1]$ be the truncated $p$-adic valuation (with $v_p(0) = 1$). Let $G$ be a truncated Barsotti-Tate group of level $n \geq 1$ non-étale over $O_K$, $G_1 = G \otimes_{O_K} (O_K/p)$. The Lie algebra of $G_1$, denoted by $\text{Lie}(G_1)$ is a finite free $O_K/p$-module. The Verschweinberg homomorphism $V_{G_1} : G_1^{(p)} \to G_1$ induces a semi-linear endomorphism $\varphi_{G_1}$ of $\text{Lie}(G_1)$. We choose a basis of $\text{Lie}(G_1)$ over $O_K/p$, and let $U$ be the matrix of $\varphi$ under this basis. We define the Hodge height of $G$, denoted by $h(G)$, to be the truncated $p$-adic valuation of $\det(U)$. We note that the definition of $h(G)$ does not depend on the choice of $U$. The Hodge height of $G$ is an analog of the Hasse invariant in mixed characteristic, and we have $h(G) = 0$ if and only if $G$ is ordinary.
Theorem 2.2 ([Fa09] Théo. 4). Let $G$ be a truncated Barsotti-Tate group of level 1 over $\mathcal{O}_K$ of dimension $d \geq 1$ and height $h$. Assume $h(G) < \frac{1}{2}$ if $p \geq 5$ and $h(G) < \frac{1}{3}$ if $p = 3$.

(i) For any rational number $\frac{e_0}{p} h(G) < a \leq \frac{e_0}{p-1} (1 - h(G))$, the finite flat subgroup $G^a$ of $G$ given by the Abbes-Saito filtration has rank $p^d$.

(ii) Let $C$ be the subgroup $G^{1 + h(G)}$ of $G$. We have $\deg(G/C) = eh(G)$.

(iii) The subgroup $C \otimes \mathcal{O}_{K,1-h(G)}$ coincides with the kernel of the Frobenius homomorphism of $G \otimes \mathcal{O}_{K,1-h(G)}$. Moreover, for any rational number $\epsilon$ with $\frac{h(G)}{p-1} < \epsilon \leq 1 - h(G)$, if $H$ is a finite and flat closed subgroup of $G$ such that $H \otimes \mathcal{O}_{K,\epsilon}$ coincides with the kernel of Frobenius of $G \otimes \mathcal{O}_{K,\epsilon}$, then we have $H = C$.

The subgroup $C$ in this theorem, when it exists, is called the canonical subgroup (of level 1) of $G$.

Remark 2.3. (i) The conventions here are slightly different from those in [Fa09]. The Hodge height is called Hasse invariant in loc. cit., while we choose to follow the terminologies in [AM04] and [Ti06]. Our index of Abbes-Saito filtration and the degree of $G$ are $\epsilon$ times those in [Fa09].

(ii) Statement (iii) of the theorem is not explicitly stated in [Fa09, Théo. 4], but it’s an easy consequence of loc. cit. Prop. 11.

For the canonical subgroups of higher level, we have

Theorem 2.4 ([Fa09] Théo. 6). Let $G$ be a truncated Barsotti-Tate group of level $n$ over $\mathcal{O}_K$ of dimension $d \geq 1$ and height $h$. Assume $h(G) < \frac{1}{2}$ if $p = 3$ and $h(G) < \frac{1}{2p^{n-1}}$ if $p \geq 5$.

(i) There exists a unique closed subgroup of $G$ that is finite and flat over $\mathcal{O}_K$ and satisfies

- $C_n(\mathcal{K})$ is free of rank $d$ over $\mathbb{Z}/p^n\mathbb{Z}$.
- For each integer $i$ with $1 \leq i \leq n$, let $C_i$ be the scheme theoretic closure of $C_n(\mathcal{K})[p^i]$ in $G$. Then the subgroup $C_i \otimes \mathcal{O}_{K,1-p^{-1}h(G)}$ coincides with the kernel of the $i$-th iterated Frobenius of $G \otimes \mathcal{O}_{K,1-p^{-1}h(G)}$.

(ii) We have $\deg(G/C_n) = \frac{e(p^n-1)}{p-1} h(G)$.

The subgroup $C_n$ in the theorem above is called the canonical subgroup of level $n$ of $G$. Fargues actually proves that $C_n$ is a certain piece of the Harder-Narasimhan filtration of $G$. The aim of this section is to show that $C_n$ appears also in the Abbes-Saito filtration.

Theorem 2.5. Let $G$ be a truncated Barsotti-Tate group of level $n$ over $\mathcal{O}_K$ satisfying the assumptions in 2.4, and $C_n$ be its canonical subgroup of level $n$. Then for any rational number $a$ satisfying $\frac{e(p^n-1)}{p-1} h(G) < a \leq \frac{e_0}{p-1} (1 - h(G))$, we have $G^a = C_n$.

Proof. We proceed by induction on $n$. If $n = 1$, the theorem is 2.2(i). We suppose $n \geq 2$ and the theorem is valid for truncated Barsotti-Tate groups of level $n - 1$. For each integer $i$ with $1 \leq i \leq n$, let $G_i$ denote the scheme theoretic closure of $G(\mathcal{K})[p^i]$ in $G$, and $C_i$ the scheme theoretic closure of $C_n(\mathcal{K})[p^i]$ in $C_n$. By Theorem 2.4(i), it’s clear that $C_i$ is the canonical subgroup of level $i$ of $G_i$. Let $a$ be a rational number with
\( \frac{ep^{n-1}}{(p-1)^2} h(G) < a \leq \frac{ep}{p-1} (1 - h(G)) \). By the induction hypothesis and the functoriality of Abbes-Saito filtration 1.3(ii), we have \( C_{n-1}(\overline{K}) = G^a_{n-1}(\overline{K}) \subset G^a(\overline{K}) \), and the image of \( G^a(\overline{K}) \) in \( G_1(\overline{K}) \) is exactly \( C_1(\overline{K}) = G^a_1(\overline{K}) \). Note that we have a commutative diagram of exact sequences of groups

\[
\begin{array}{ccccccc}
0 & \longrightarrow & C_{n-1}(\overline{K}) & \longrightarrow & C_n(\overline{K}) & \longrightarrow & C_1(\overline{K}) & \longrightarrow & 0 \\
0 & \longrightarrow & G_{n-1}(\overline{K}) & \longrightarrow & G(\overline{K}) & \longrightarrow & C_1(\overline{K}) & \longrightarrow & 0,
\end{array}
\]

where vertical arrows are natural inclusions. So we have \( C_n(\overline{K}) \subset G^a(\overline{K}) \). On the other hand, Theorems 0.1 and 2.4(ii) imply that \( (G/C_n)^a(\overline{K}) = 0 \) as \( a > \frac{ep^{n-1}}{(p-1)^2} h(G) = \frac{p}{p-1} \text{deg}(G/C_n) \). Therefore, we get \( G^a(\overline{K}) \subset C_n(\overline{K}) \) by 1.3(ii). This completes the proof.

\[ \square \]

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